Eighth Edition

CHAPTER

5

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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Lecture Notes:

J. Walt Oler

Texas Tech University

نیروهای گسترده:

مرکز سطح و مرکز جرم

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مقدمه

- زمین به اجسام نیروی جاذبه وارد می‌کند. این نیرو می‌تواند با یک نیروی معادل که در **مرکز جرم** جسم وارد می‌شود جایگزین شود.

- مرکز سطح یک سطح همان مرکز جرم آن سطح است. مفهوم گشتاور اول یک سطح به منظور پیدا کردن این نقطه مطرح شده است.

- تئوری پاپوس ـ گلیدینوس به منظور محاسبه سطح دوران یا حجم دوران یک جسم مورد استفاده قرار می‌گیرد.

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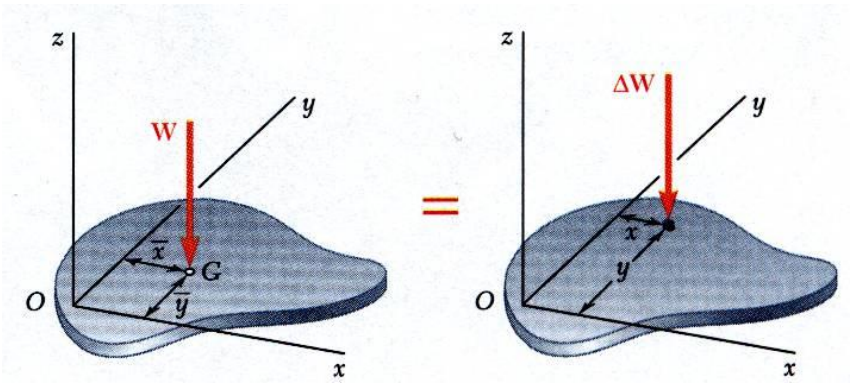
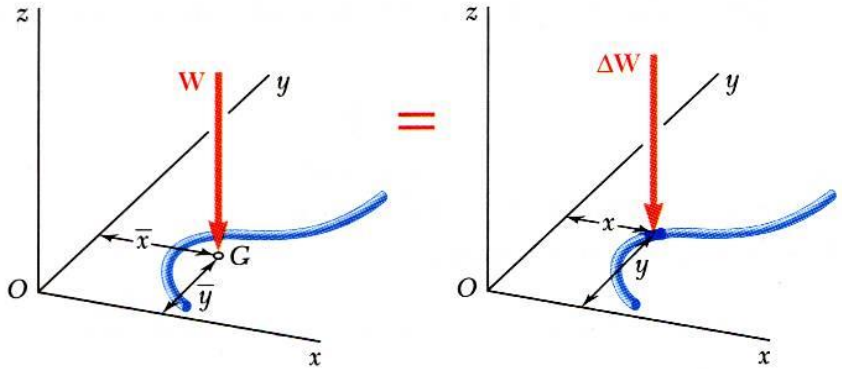
مرکز جرم اجسام دوبعدی

- مرکز جرم یک سیم
- مرکز جرم یک ورق





$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$

$$= \int x\,dW$$

$$\sum M_y \quad \bar{y}W = \sum y\Delta W$$

$$= \int y\,dW$$

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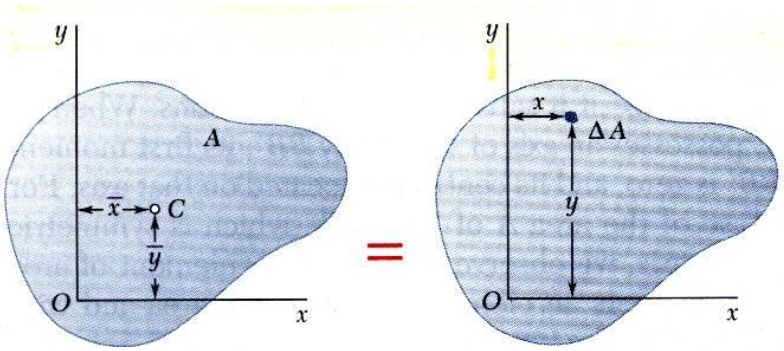
مرکز سطح و گشتاور اول خطوط و سطوح

- مرکز سطح یک سطح



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t)dA$$

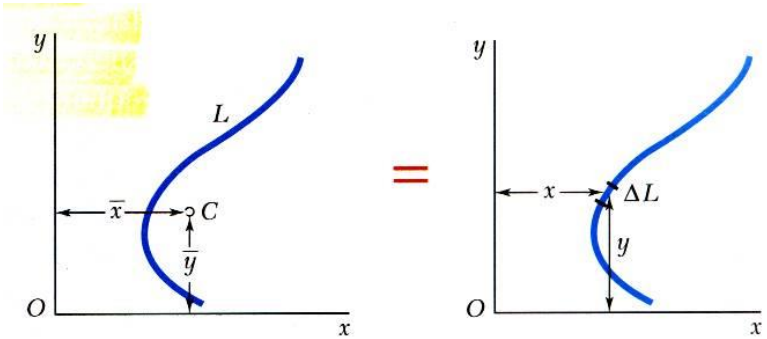
$$\bar{x}A = \int x\,dA = Q_y$$

$$= \text{first moment with respect to } y$$

$$\bar{y}A = \int y\,dA = Q_x$$

$$= \text{first moment with respect to } x$$

- مرکز سطح یک منحنی



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma L a) = \int x(\gamma a)dL$$

$$\bar{x}L = \int x\,dL$$

$$\bar{y}L = \int y\,dL$$

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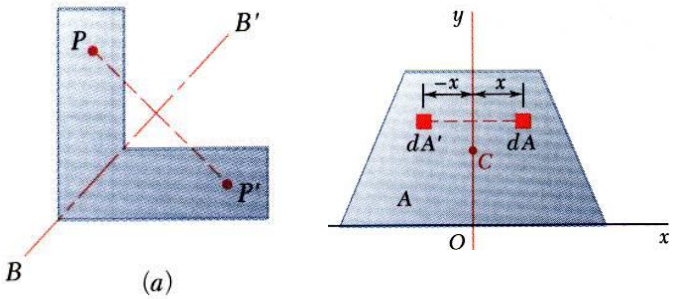
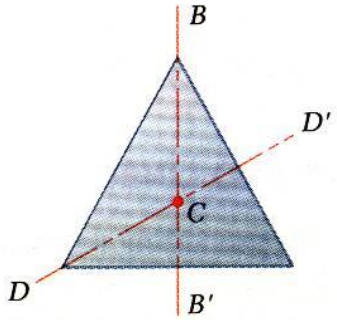
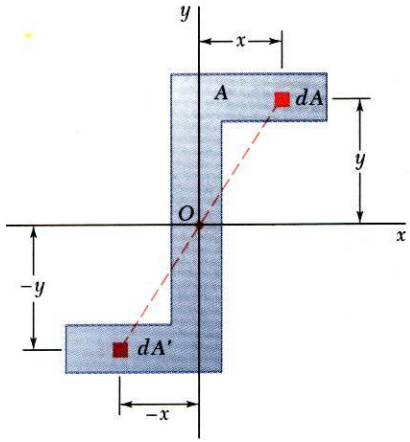
گشتاور اول سطوح و خطوط



- محور BB’ محور تقارن یک سطح است در صورتی که با در نظر گرفتن هر نقطه P، نقطه P’ موجود باشد که خط PP’ بر محور BB’ عمود بوده و این محور PP’ را به دو قسمت مساوی تقسیم کند.
- گشتاور اول سطح حول محور تقارن برابر صفر است.



- اگر یک سطح محور تقارن داشته باشد، مرکز سطح حتماً روی این محور قرار خواهد گرفت.
- اگر یک سطح دو محور تقارن داشته باشد، مرکز سطح محل برخورد این دو محور خواهد بود.



- یک سطح نسبت به نقطه ای مانند O متقارن خواهد بود اگر برای هر جزء سطح dA در نقطه (x, y) جزء سطحی به همان اندازه در نقطه (-x, -y) وجود داشته باشد.
- مرکز سطح یک جسم منطبق بر مرکز تقارن آن خواهد بود.

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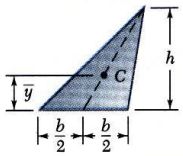
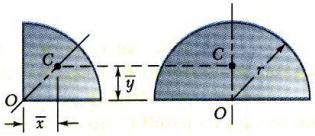
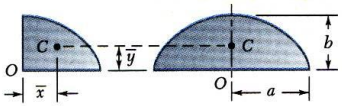
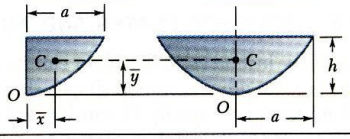
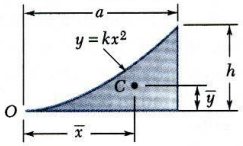
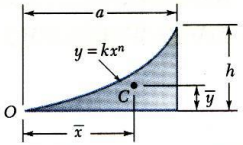
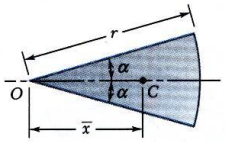
مرکز سطح سطوح متداول

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

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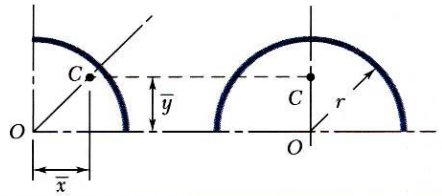
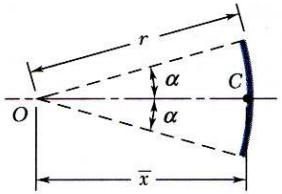
مرکز سطح سطوح متداول

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

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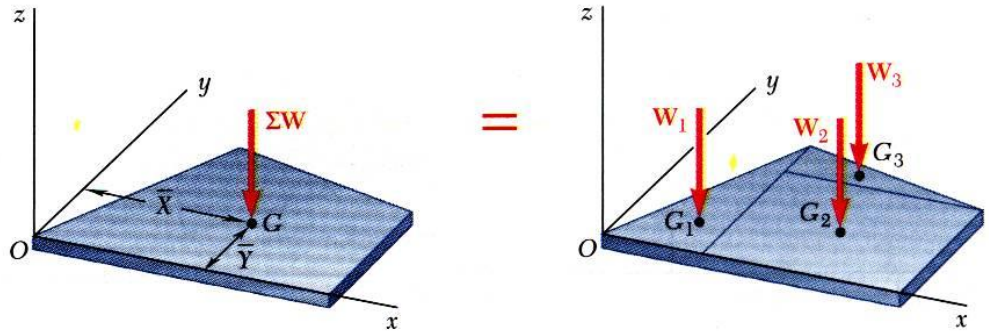
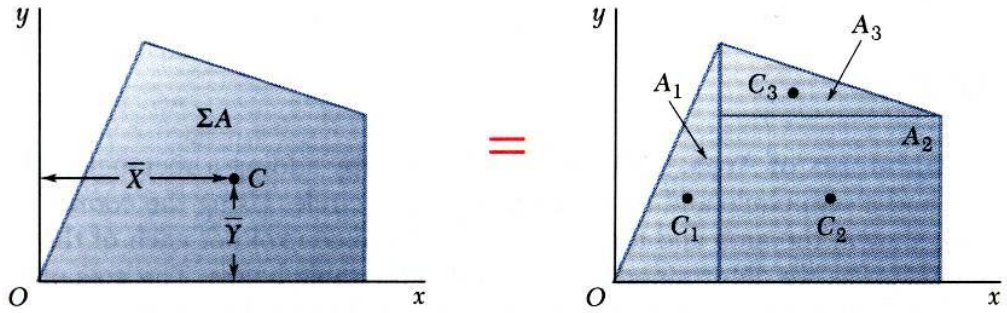
ورق‌ها و سطوح چندجزئی



- ورقهای چندجزئی

$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



- سطوح چندجزئی

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$

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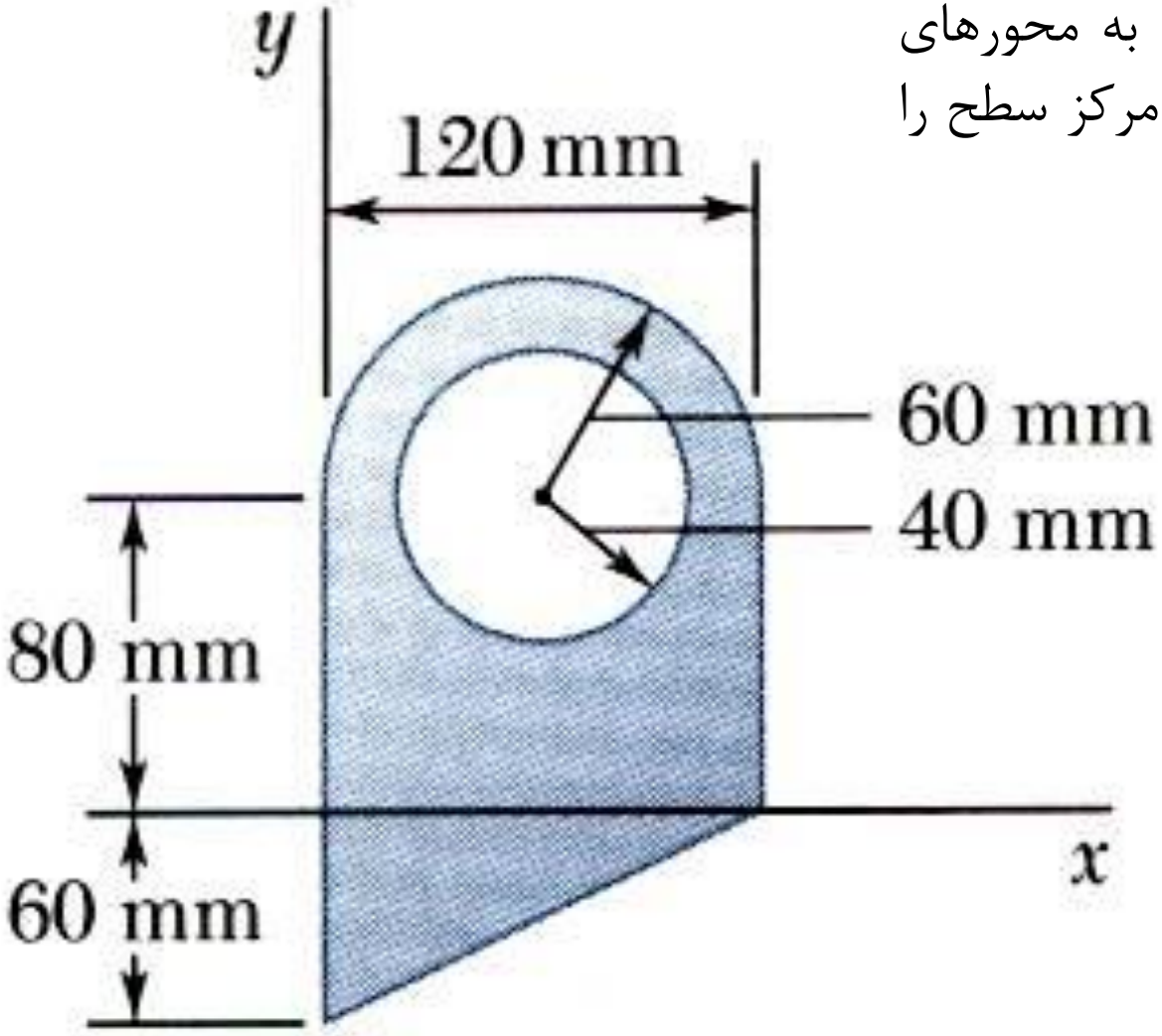
مسئله نمونه ۵–۱

گشتاور اول سطح را نسبت به محورهای x و y تعیین کرده و محل مرکز سطح را تعیین نمایید.



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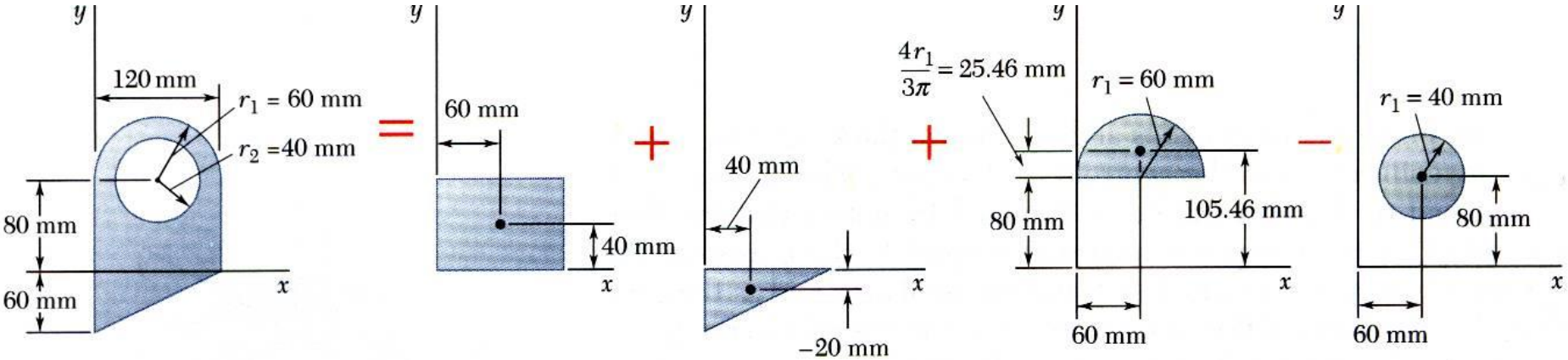
مسئله نمونه ۵–۱



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$

$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

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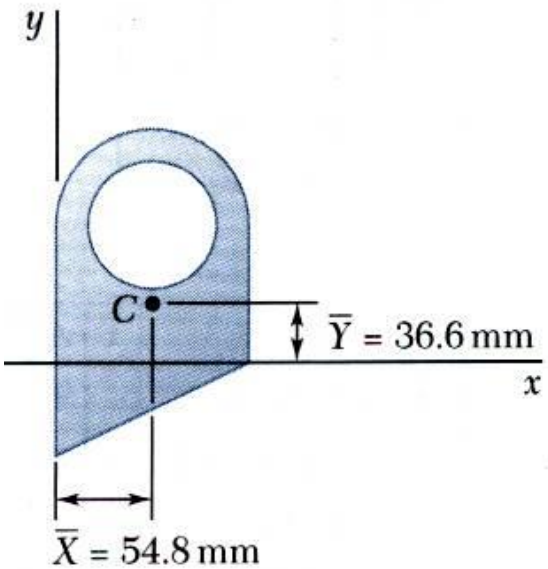
مسئله نمونه ۵–۱



$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\boxed{\bar{X} = 54.8\,\text{mm}}$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\boxed{\bar{Y} = 36.6\,\text{mm}}$$

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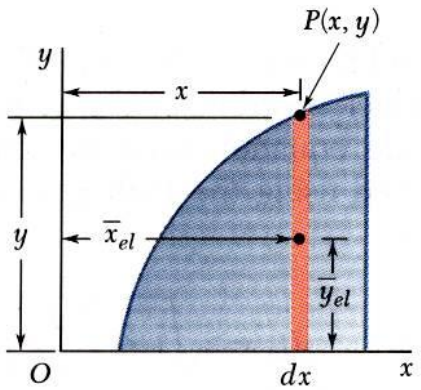
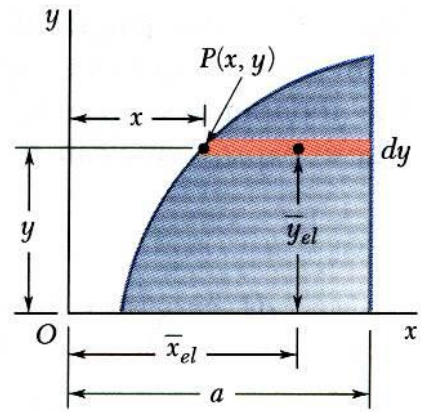
تعیین مرکز سطح با انتگرال‌گیری

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$

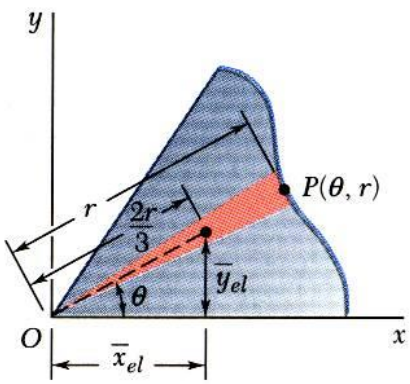
$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$



$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int x\,(y\,dx)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{y}{2}(y\,dx)$$



$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{a+x}{2}\left[(a-x)dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int y\left[(a-x)dx\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)$$

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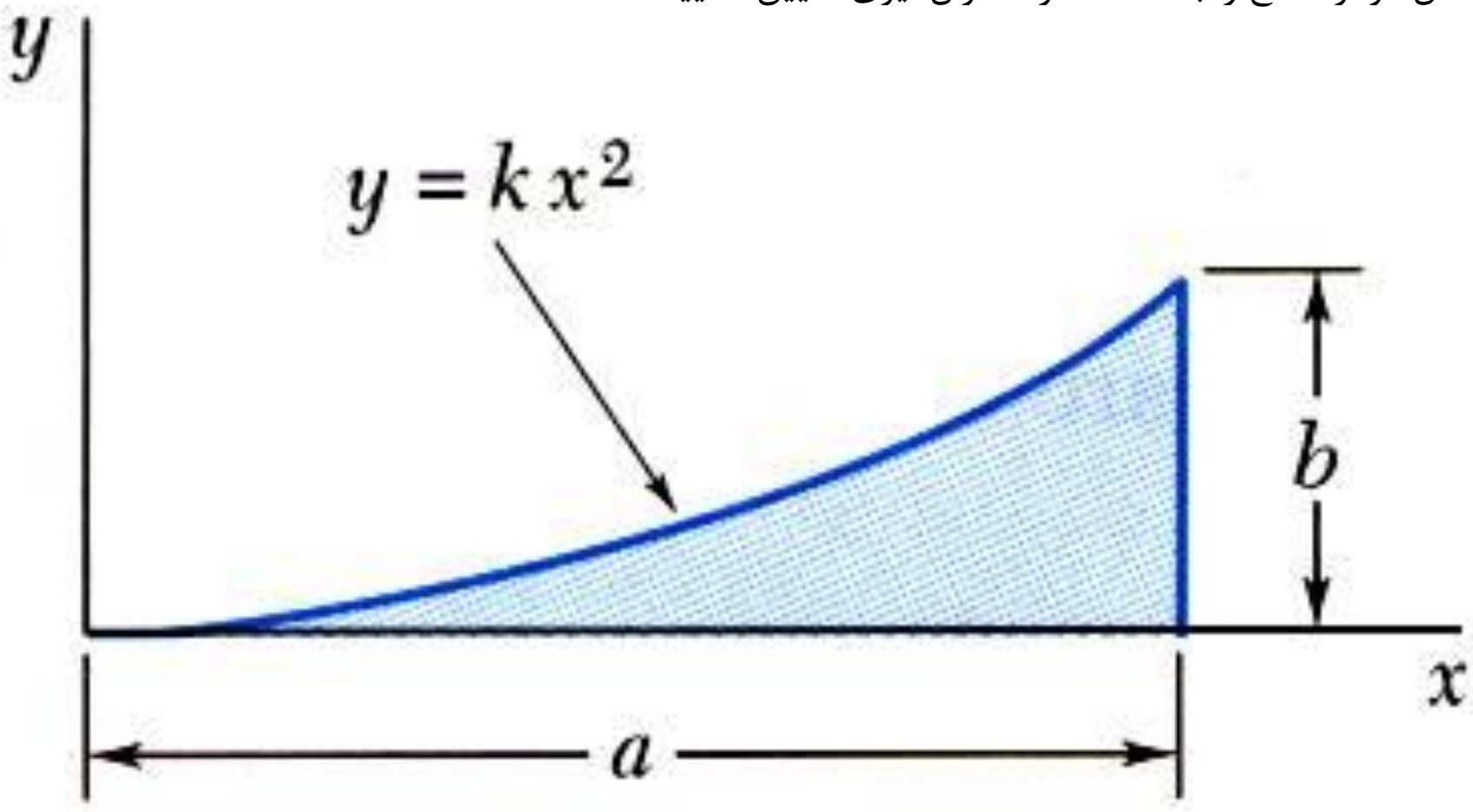
مسئله نمونه ۵–۲

محل مرکز سطح را با استفاده از انتگرال‌گیری تعیین نمایید.



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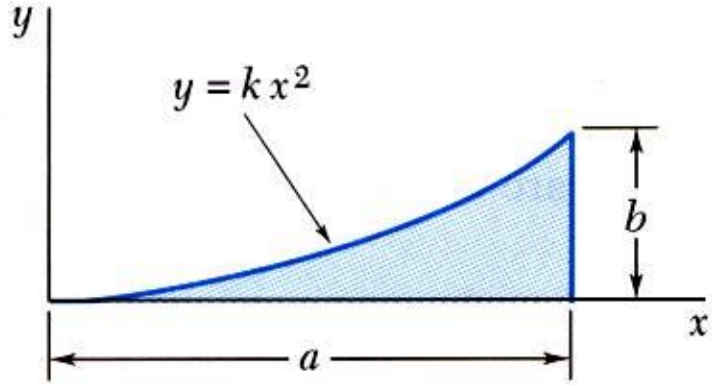
مسئله نمونه ۵–۲

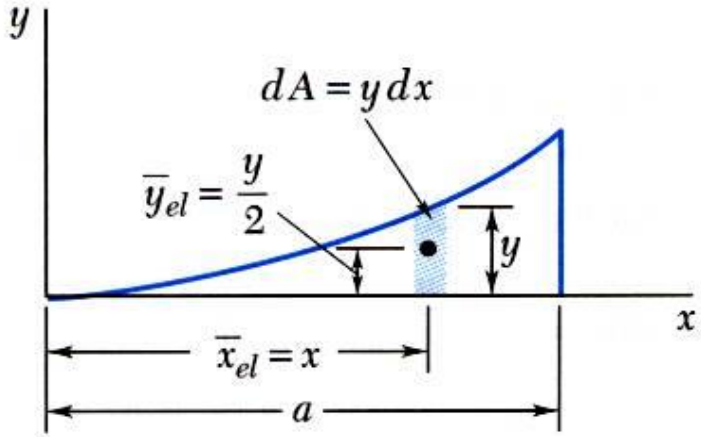


- تعیین مقدار ثابت k

$$y = k x^2$$

$$b = k a^2 \Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2} x^2 \quad or \quad x = \frac{a}{b^{1/2}} y^{1/2}$$



- تعیین مساحت کل

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2} x^2 dx = \left[\frac{b}{a^2} \frac{x^3}{3} \right]_0^a$$

$$= \frac{ab}{3}$$

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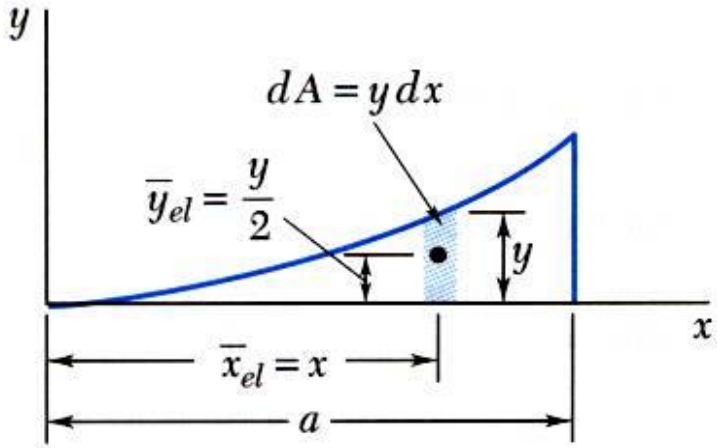
مسئله نمونه ۵–۲



$$Q_y = \int \bar{x}_{el}\, dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int \frac{y}{2}\, y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

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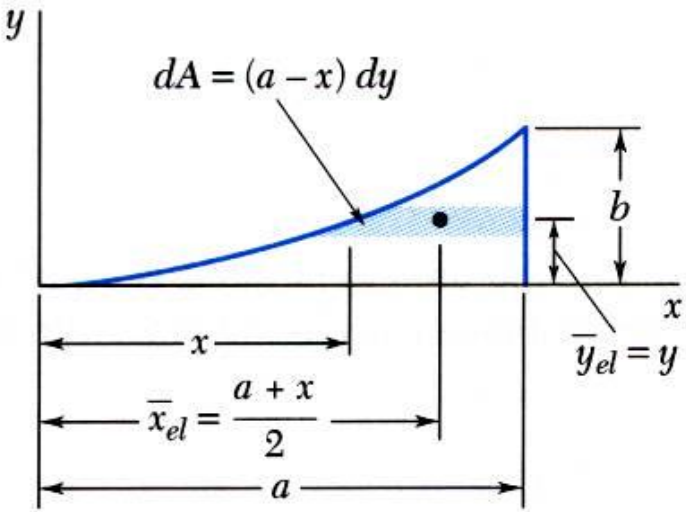
مسئله نمونه ۵–۲



$$Q_y = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int y(a-x)dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

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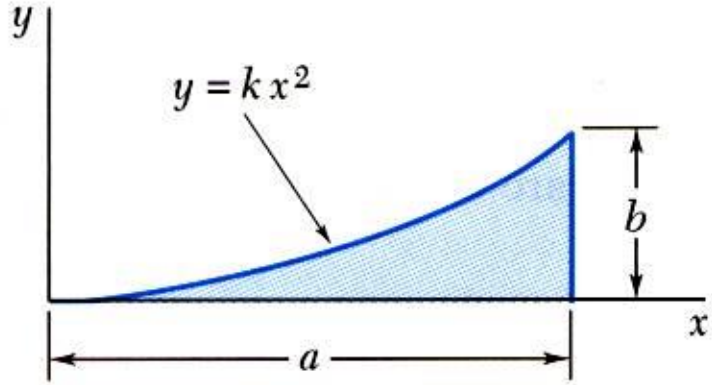
مسئله نمونه ۵–۲



$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4}$$

$$\boxed{\bar{x} = \frac{3}{4}a}$$

$$\bar{y}A = Q_x$$

$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\boxed{\bar{y} = \frac{3}{10}b}$$

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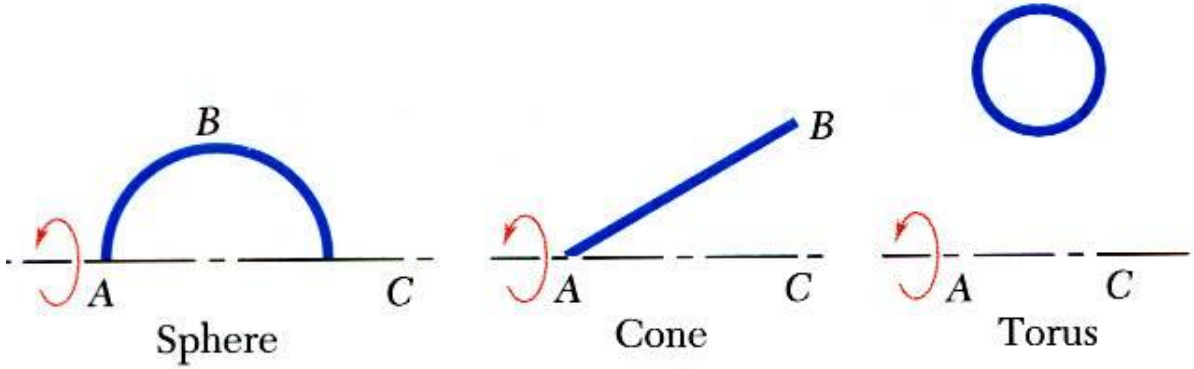
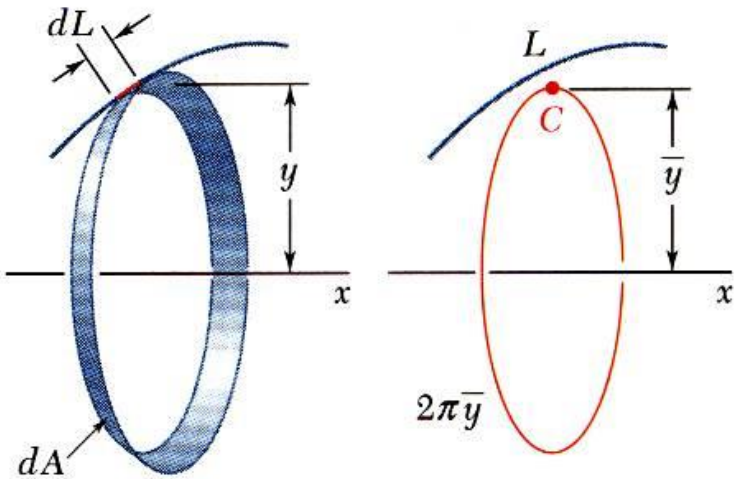
تئوری پاپوس ــ گلدینوس



- سطح دوران با چرخش منحنی حول یک محور ایجاد می‌شود.



- مساحت سطح دوران برابر با مسافتی است که سطح حول محور طی نموده است.

$$A = 2\pi\bar{y}L$$

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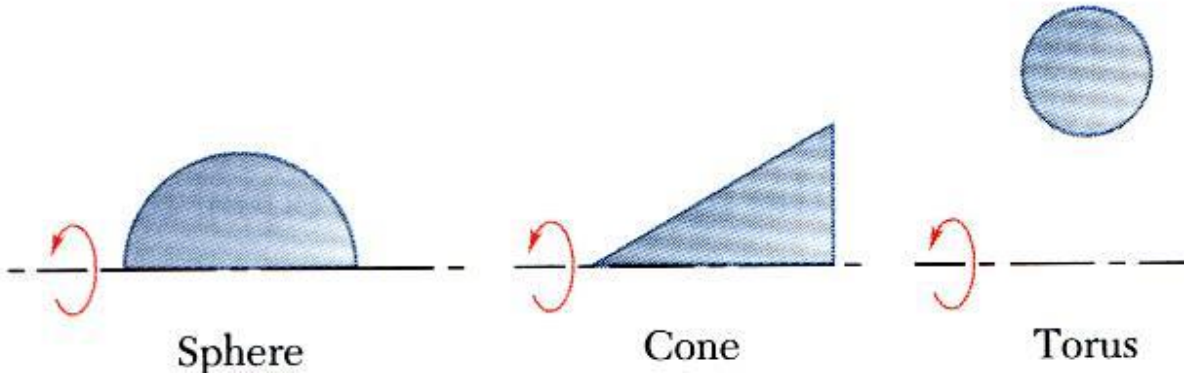
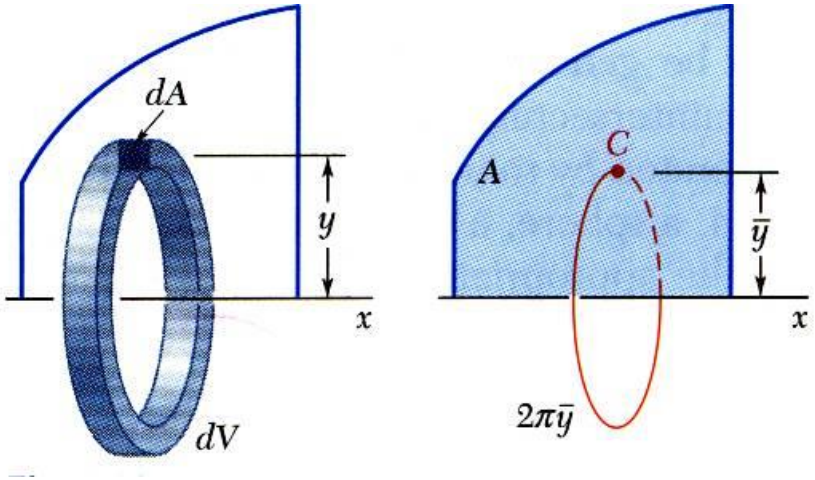
تئوری پاپوس ـ گلدینوس



- حجم دوران با دوران یک سطح حول یک محور ایجاد می‌شود.



- حجم ایجاد شده برابر با مقدار مسافتی است که یک سطح حول محور چرخیده است.

$$V = 2\pi\bar{y}A$$

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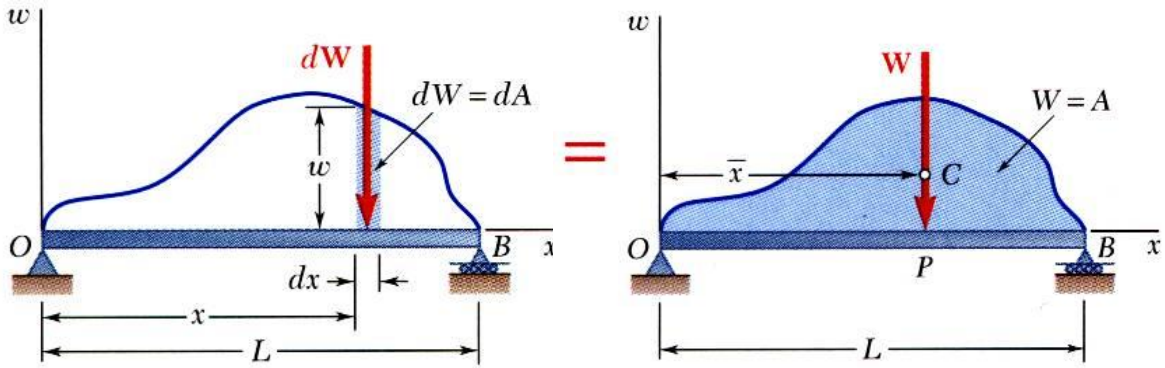
بارهای گسترده روی تیرها



$$W = \int_0^L w\,dx = \int dA = A$$

- یک بار گسترده میتواند با یک بار متمرکز معادل جایگزین شود که مقدار این بار برابر حاصلضرب اندازه بار در سطح آن و محل اعمال آن در مرکز سطح بار وارده است.

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

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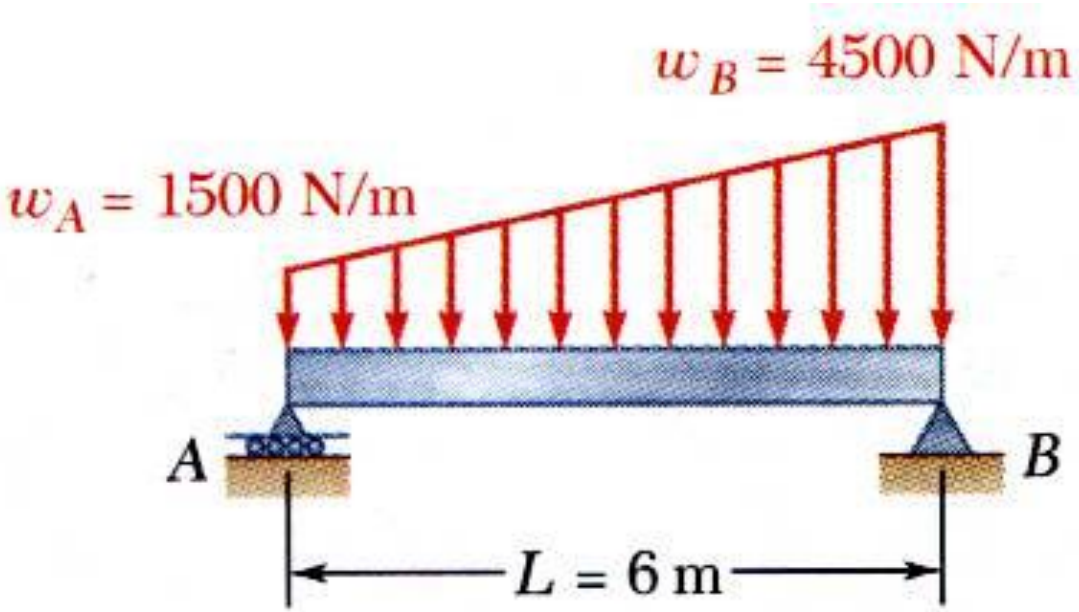
مسئله نمونه ۵–۳

مقدار بار متمرکز، محل اعمال آن و عکس‌العمل‌های تکیه‌گاهی را تعیین کنید.



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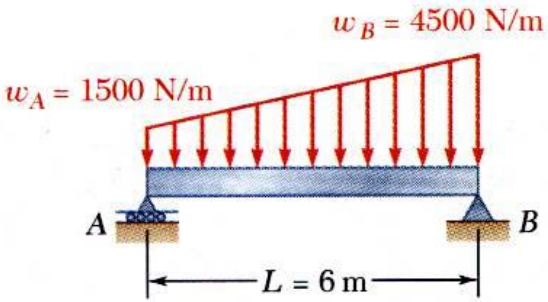
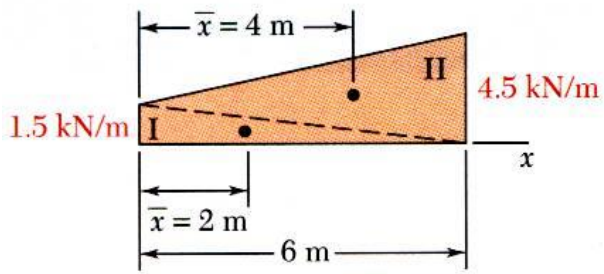
مسئله نمونه ۵–۳

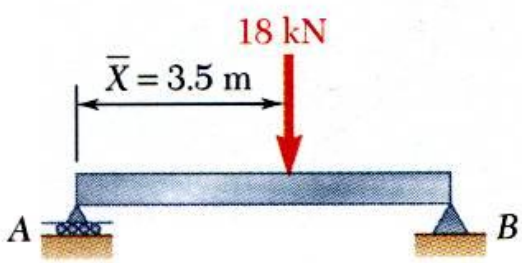


$$F = 18.0\ \text{kN}$$



$$\bar{X} = \frac{63\ \text{kN}\cdot\text{m}}{18\ \text{kN}}$$

$$\bar{X} = 3.5\ \text{m}$$



Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

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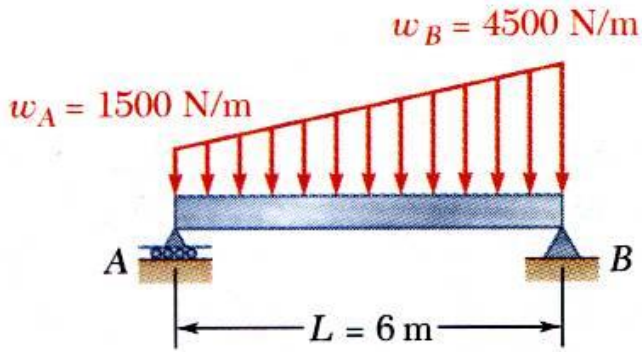
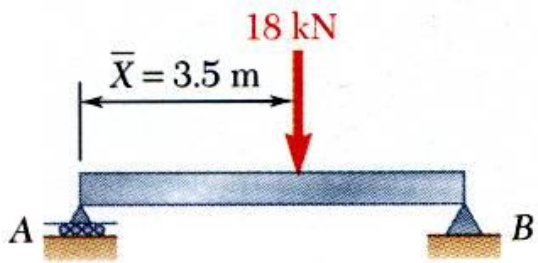
مسئله نمونه ۵–۳



$$\sum M_A = 0: \quad B_y(6\,\text{m}) - (18\,\text{kN})(3.5\,\text{m}) = 0$$

$$\boxed{B_y = 10.5\,\text{kN}}$$



$$\sum M_B = 0: \quad -A_y(6\,\text{m}) + (18\,\text{kN})(6\,\text{m} - 3.5\,\text{m}) = 0$$

$$\boxed{A_y = 7.5\,\text{kN}}$$

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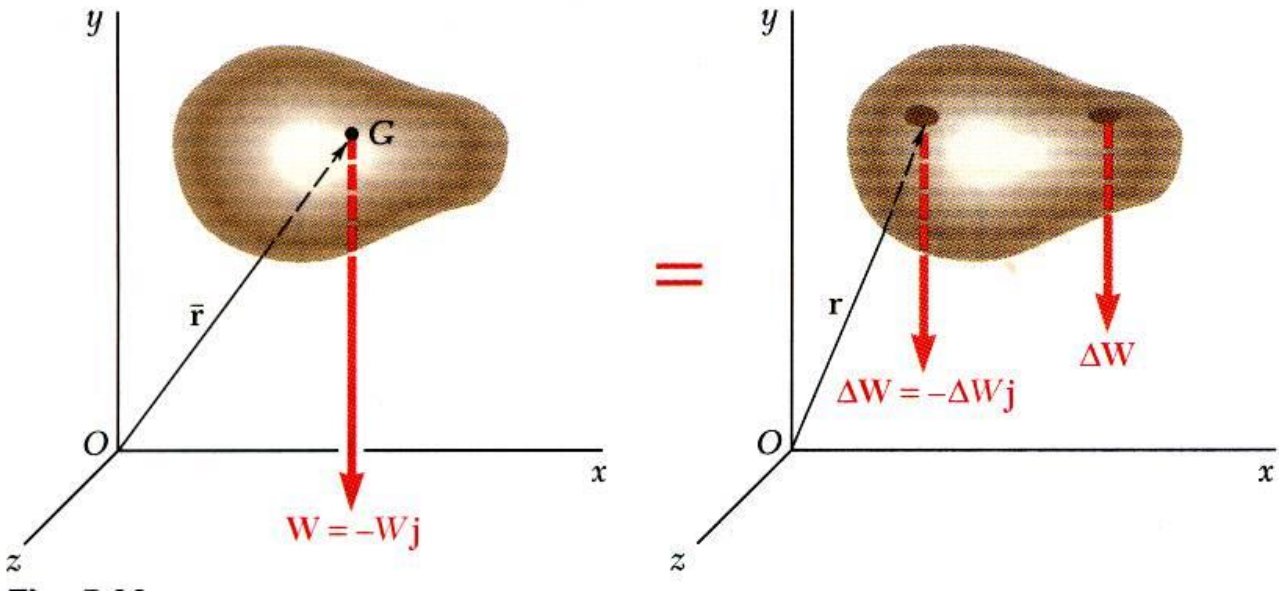
مرکز حجم و جرم اجسام سه بعدی



- نتایج مستقل از جهت گیری جسم در فضا است

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW$$

- برای اجسام همگن

$$W = \gamma V \quad \text{and} \quad dW = \gamma\, dV$$

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV$$

- مرکز جرم

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times \left(-W\vec{j}\right) = \sum\left[\vec{r} \times \left(-\Delta W\vec{j}\right)\right]$$

$$\vec{r}_G W \times \left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right) \times \left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\,dW$$

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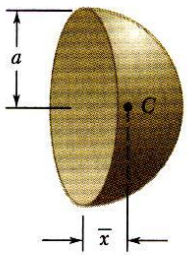
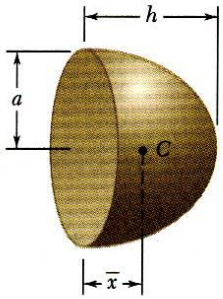
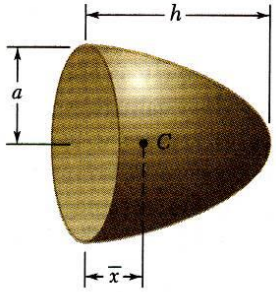
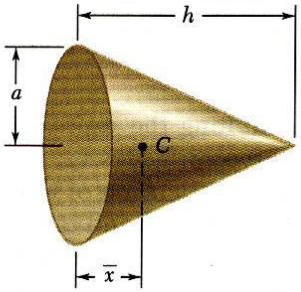
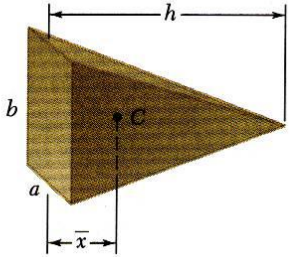
مرکز حجم اجسام متداول

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

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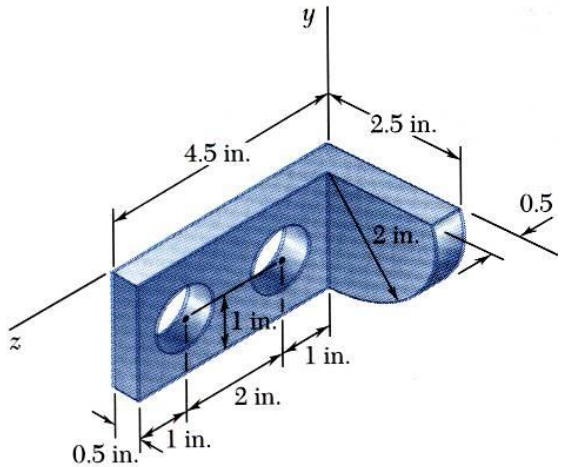
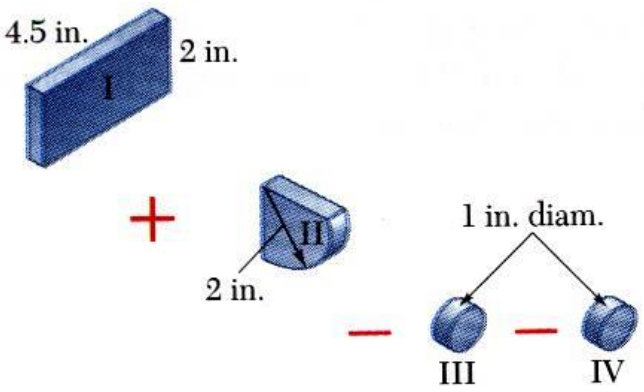
ترکیب اجسام سه بعدی



$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

- برای اجسام همگن

$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$



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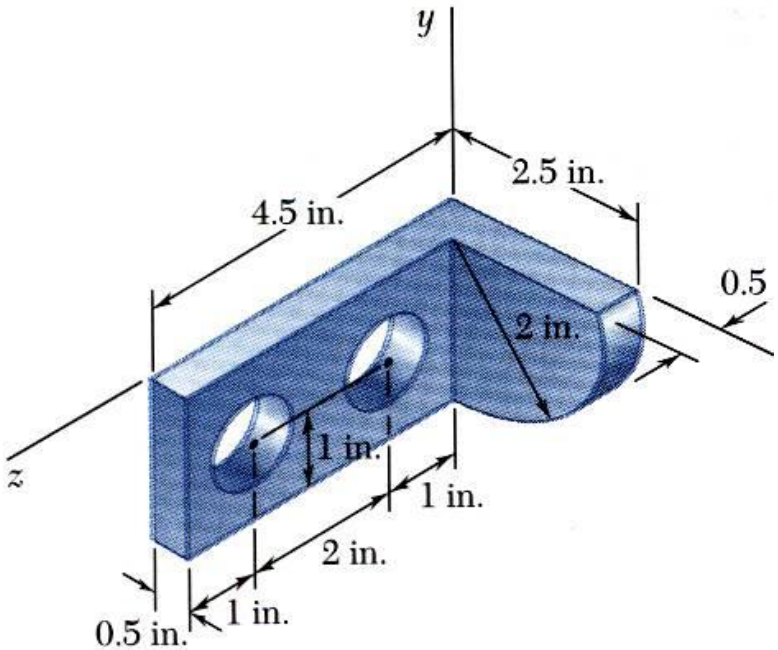
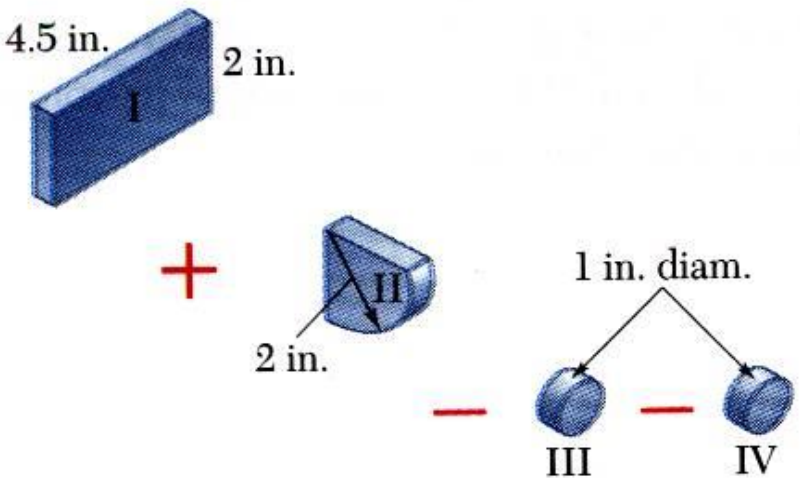
مسئله نمونه ۵–۴

مرکز جرم حجم زیر را تعیین کنید. قطر هر یک از دایره ها یک اینچ است.





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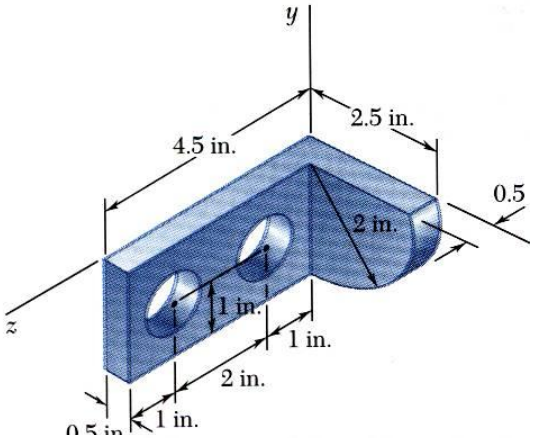
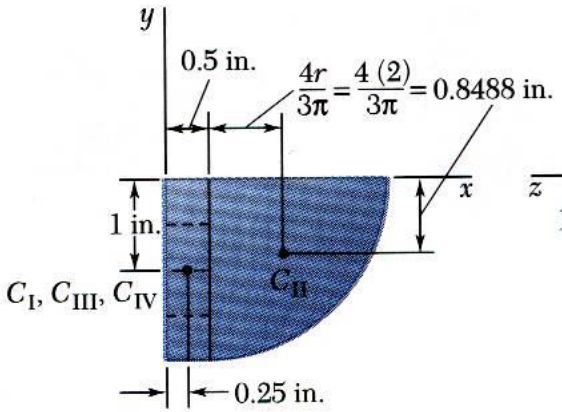
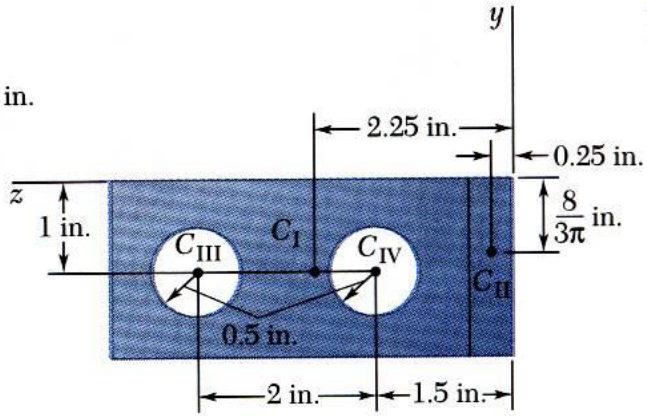
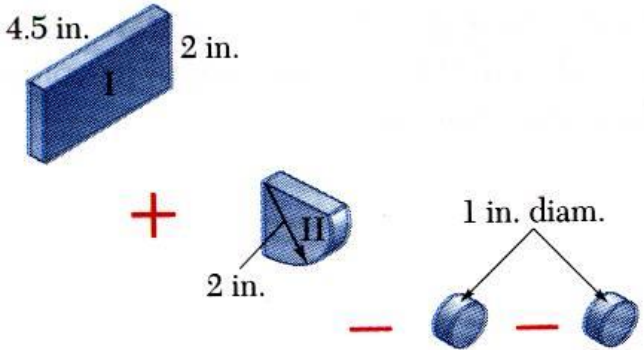
مسئله نمونه ۵–۴









	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

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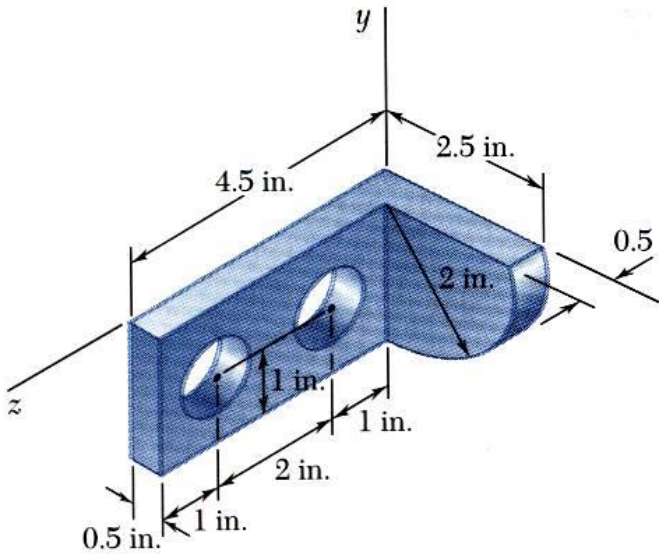
مسئله نمونه ۵–۴

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$



$$\bar{X} = \sum \bar{x}V / \sum V = \left(3.08\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{X} = 0.577\,\text{in.}}$$

$$\bar{Y} = \sum \bar{y}V / \sum V = \left(-5.047\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{Y} = 0.577\,\text{in.}}$$

$$\bar{Z} = \sum \bar{z}V / \sum V = \left(1.618\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{Z} = 0.577\,\text{in.}}$$

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