حل مسائلی برگرفته از کتاب انتقال حرارت هدایتی

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انتشارات جهاددانشگاهی واحد تهران

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«ویزکیهم و یعلمهم الکتاب و الحکمة»

مقدمه ناشر

ایران امروز در اشتیاق توسعه و استقلال، گامهای محکم و استواری برمی‌دارد. همه روزه در گوشه و کنار میهن ما جوانه‌های خود کفائی علمی و فنی رخ نموده و با عنایت و یاری خداوند متعال و در سایه تلاش و کوشش جامعه علمی و دانشگاهی حرکت به سوی مرزهای دانش شتاب بیشتری به خود می‌گیرد. خدای را شکر می‌گوییم که این فرصت را به ما ارزانی داشته تا گامهایی هر چند کوچک در راه رشد و نشر دستاوردهای علمی و فرهنگی کشور برداریم، باشد تا با یاری خداوند منان و در پرتو همت اندیشمندان، نویسندگان، مترجمان و متخصصان مومن و متعهد بتوانیم در اعتلاء علمی کشور عزیزمان ایران سهمی داشته باشیم.

انتشارات جهاد دانشگاهی دانشگاه تهران در راستای وظایف خویش و به منظور نیل به اهداف علمی– فرهنگی نظام جمهوری اسلامی مبادرت به انتشار آثار ارزشمند و مورد نیاز علمی و دانشگاهی می‌نماید. در این راه از کلیه اساتید، پژوهشگران، صاحبان قلم و اندیشه دعوت به مشارکت و همکاری می‌گردد.

پیشگفتار مولف

کتاب انتقال حرارت هدایتی آرپاچی بدون شک یکی از بهترین منابع فرمولاسیون سیستم‌های دارای انتقال حرارت هدایتی است و در بسیاری از دانشگاه‌های دنیا در رشته‌های مهندسی شیمی و مهندسی مکانیک تدریس می‌شود. در انتهای هر فصل از این کتاب با ارزش تعدادی سوال به عنوان تمرین به خواننده واگذار شده است که حل این مسائل کمک زیادی به درک عمیق‌تر و بهتر مطالب کتاب می‌نماید زیرا از طرفی خواننده با نحوه فرمولاسیون یک سیستم آشنا شده و از طرف دیگر نحوه حل تحلیلی مسائل فرموله شده را فرا می‌گیرد.

با توجه به اهمیت مسائل آخر فصل کتاب انتقال حرارت آرپاچی و همچنین فقدان مرجعی که بتوان پاسخ‌های صحیح را از آن استخراج نمود، بر آن شدیم که کتاب حاضر را در اختیار علاقه‌مندان قرار دهیم. در نگارش این کتاب از برخی مسائل فصول ۲، ۳، ۴، و ۵ کتاب انتقال حرارت هدایتی آرپاچی استفاده شده است زیرا مطالب این فصول تقریباً در همه دانشگاه‌هایی که از کتاب انتقال حرارت هدایتی آرپاچی استفاده می‌نمایند، تدریس می‌شود.

بدون شک حل کامل و بدون اشکال تمامی مسائل این کتاب اگرچه بعید نیست ولی کاری دشوار و بسیار زمان‌بر است. لذا از تمامی اساتید، دانشجویان، مهندسان، و خوانندگان عزیز تقاضا می‌نماییم در تکمیل و بهبود این اثر یاری‌مان نمایند و نظرات و پیشنهادات خود و اشکالات موجود در کتاب را توسط پست الکترونیکی با مولفان کتاب در میان گذارند.

با آرزوی موفقیت و با سپاس‌گذاری فراوان

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بهار ۱۳۹۰

فهرست مطالب

فصل دوم

فرمولاسیون‌های متمرکز، انتگرالی و دیفرانسیلی

مساله ۱-۲)

با تخلیه مایع در حجم ثابت فشار مخزن کم خواهد شد

a)

$m_L =$ جرم مایع، $v_L =$ حجم مخصوص مایع

$m_g =$ جرم بخار، $v_g =$ حجم مخصوص بخار

$m_1 =$ جرم کل مخلوط در حالت ۱ و $m_2 =$ جرم کل مخلوط در حالت ۲

$$m_2 = m_1 - 1 \;\Rightarrow\; m_{g2} + m_{l2} = m_{g1} + m_{l1} - 1$$

سیستم بسته است: $V_1 = V_2 \;\Rightarrow\; m_{g1}v_g + m_{l1}v_L = m_{g2}v_g + m_{l2}v_L$

$$\Rightarrow\; (m_{g2} - m_{g1})v_g = (m_{l1} - m_{l2})v_L = -(m_{l2} - m_{l1})v_L$$

$$\Rightarrow\; (m_{g2} - m_{g1})v_g + \left(1 + (m_{g2} - m_{g1})\right)v_L = 0$$

$$\Rightarrow\; (m_{g2} - m_{g1})(v_g - v_L) = v_L \;\Rightarrow\; \frac{v_L}{v_g - v_L} = (m_{g2} - m_{g1})$$

$$\Rightarrow\; \frac{v_L v_g}{v_g - v_L} = (m_{g2} - m_{g1})v_g = m_{g2}v_g - m_{g1}v_g = V_{g2} - V_{g1} = \Delta V_g$$

$$\Rightarrow\; \Delta V_g = \frac{v_L v_g}{v_g - v_L}$$

b)

برطبق قانون دوم ترمودینامیک:

$$Q - W = \Delta U$$

If $V=$ cons $\Rightarrow W = 0 \Rightarrow Q = \Delta U = U_1 - U_2$

دما ثابت است بنابراین ویژگی های ترمودینامیکی نیز ثابت می ماند:

$$\begin{cases} U_2 = m_{l2}u_l + m_{g2}u_g \\ U_1 = m_{l1}u_l + m_{g1}u_g \end{cases}$$

$$Q = \Delta U = U_1 - U_2 = m_2u_2 - m_1u_1$$

$$= m_{g2}u_g + m_{l2}u_l - m_{g1}u_g - m_{l1}u_l$$

$$Q = (m_{g2} - m_{g1})u_g + (m_{l2} - m_{l1})u_l$$

$$= (m_{g2} - m_{g1})u_g + \left(-1 - (m_{g2} - m_{g1})\right)u_l$$

$$Q = (m_{g2} - m_{g1})(u_g - u_l) - u_l = \frac{v_L}{v_g - v_L}(u_g - u_L) - u_L$$

$$= \frac{v_L u_g - v_L u_L - u_L v_g + v_L u_L}{v_g - v_L} \Rightarrow Q = \frac{v_L u_g - u_L v_g}{v_g - v_L}$$

مساله ۲-۲)

از آنجایی‌که $T_W \gg T_o$ است، تشعشع خواهیم داشت و با ناچیز فرض کردن اصطکاک، برطبق قانون دوم ترمودینامیک خواهیم داشت:

$$F = m \cdot a \Rightarrow W = m\frac{d^2y}{dt^2} \Rightarrow m_g = m\frac{d^2y}{dt^2} \Rightarrow y = \frac{1}{2}gt^2 + C_1t + C_2$$

At $t = 0 \Rightarrow V = \frac{dy}{dx} = 0 \Rightarrow C_1 = 0$

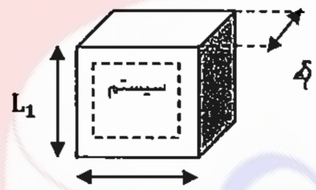
At $t = 0 \Rightarrow y = 0 \Rightarrow C_2 = 0$

$$\Rightarrow y = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2(L_2 - L_1)}{g}} \Rightarrow t_{end} = \sqrt{\frac{2(40-0.5)}{32.2}} =$$

1.5663 Sec

بنابراین بعد از 1.5663 ثانیه صفحه به ته محفظه می‌رسد.

برطبق موازنه انرژی و قانون استفان- بولتزمان خواهیم داشت:



$$A_1\sigma\bar{F}_{12}(T_W^4 - T^4) = L_1^2\delta\,\rho_1c_1\frac{dT}{dt} = q_{12}\cdot A_1$$

$$q_{12} = \sigma\bar{F}_{12}(T_W^4 - T^4)$$

$$\frac{1}{\bar{F}_{12}} = \frac{1-\epsilon_1}{\epsilon_1A_1} + \frac{1}{A_2F_{21}} + \frac{1-\epsilon_2}{\epsilon_2A_2}$$

$$A_1 = 2L_1^2 + 4\delta L_1 = 2\times0.5^2 + 4\times\frac{1}{48}\times0.5 = 0.5417\ \text{ft}^2$$

$$A_2 = \pi D_2L_2 + 2\pi\frac{D_2^2}{4} = 3.14\times10\times40 + 3.14\times10^2\times0.5 = 1413\ \text{ft}^2,$$

$$F_{12} = 1 \Rightarrow A_1F_{12} = A_2F_{21} = 1\times0.5417 = 0.5417$$

$$\Rightarrow \frac{1}{\bar{F}_{12}} = \frac{1-0.8}{0.8\times0.5417} + \frac{1}{0.5417} + \frac{1-0.4}{0.4\times1413} = 2.308 \Rightarrow \bar{F}_{12} = 0.433$$

$$q_{12}\cdot A_1 = A_1\sigma\bar{F}_{12}(T_W^4 - T^4) = L_1^2\delta\,\rho_1c_1\frac{dT}{dt} \Rightarrow$$

$$\frac{dT}{dt} = \frac{A_1\sigma\bar{F}_{12}}{L_1^2\delta\,\rho_1c_1}(T_W^4 - T^4) = \alpha(T_W^4 - T^4) = \alpha T_W^4 - \alpha T^4 = \beta - \alpha T^4$$

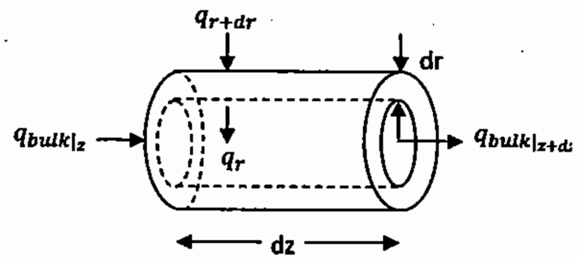
$$\alpha = \frac{A_1\sigma\bar{F}_{12}}{L_1^2\delta\,\rho_1c_1} = 4.2532\times10^{-13}\left(\frac{1}{Sec\cdot R^3}\right)$$

$$\beta = \alpha T_W^4 = 6.805\left(\frac{°R}{Sec}\right)$$

$$I = \int_{T_o}^{T_{end}}\frac{dT}{\beta - \alpha T^4} = \int_o^{t_{end}}dt \Rightarrow \int_{60+459.67}^{T_{end}}\frac{dT}{\beta - \alpha T^4} = t_{end} = 1.5663\ Sec$$

مساله ۳-۲)

ابتدا یک المان استوانه‌ای را در نظر می‌گیریم و سپس برای آن موازنه انرژی می‌نویسیم:



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

در جهت z:

$$\dot{E}_{in} = \dot{m}\bar{c}T = \rho V\bar{c}T\,|_Z = \rho(2\pi rdr)\bar{c}T\,|_Z$$

$$\dot{E}_{out} = \dot{m}\bar{c}T = \rho V\bar{c}T\,|_{Z+dZ} = \rho(2\pi rdr)\bar{c}T\,|_{Z+dZ}$$

در جهت r: $\dot{E}_{in} = q_{r+dr} \times A_{r+dr} = \left(k\,\frac{dT}{dr}\right)(2\pi r dz)|_{r+dr}$

$$\dot{E}_{out} = q_r \times A_r = \left(k\,\frac{dT}{dr}\right)(2\pi r dz)|_{r}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{V_r}{\alpha}\frac{\partial T}{\partial z} = 0$$

$$\mathrm{BC}\begin{cases} r = 0 \;\Rightarrow\; \dfrac{\partial T}{\partial r} = 0 \\ r = \dfrac{D}{2} \;\Rightarrow\; q'' = k\dfrac{\partial T}{\partial r} \\ Z = L \;\Rightarrow\; T = T(r) \end{cases} \quad \int_l^L \int_0^R \left[\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \rho c V_r \frac{\partial T}{\partial z}\right] dr\,dz = 0$$

$$\Rightarrow \int_l^L \int_0^R \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{V_r}{\alpha}\frac{\partial T}{\partial z}\right] dr\,dz = 0$$

قبل از حل مساله ابتدا پروفیل سرعت را به دست می‌آوریم، در صورت مساله اشاره شده که این پروفیل سهموی است:

$$V_r = C_1 r^2 + C_2 r + C_3$$

$$r = 0 \Rightarrow \frac{\partial V}{\partial r} = 0 \Rightarrow C_2 = 0$$

$$r = \frac{D}{2} = R \Rightarrow V = 0 \Rightarrow C_3 = -C_1 R^2$$

$$\Rightarrow V_r = C_1(r^2 - R^2)$$

سرعت متوسط $= V = \int_0^R V_r(2\pi r\,dr) = 2\pi C_1 \int_0^R (r^3 - rR^2)dr =$

$$2\pi C_1\left(\frac{R^4}{4} - \frac{R^4}{2}\right) = \frac{-R^4\pi C_1}{2} \Rightarrow C_1 = \frac{-2V}{R^2}$$

$$\Rightarrow V_r = \frac{-2V}{R^2}(r^2 - R^2) = 2V\left(1 - \left(\frac{r}{R}\right)^2\right) = 2V\left(1 - \left(\frac{2r}{D}\right)^2\right) =$$

$$V_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

سپس برای دما خواهیم داشت:

$$T_{z,r} = (C_4 r^2 + C_5 r + C_6)Z(z)$$

$$r = 0 \Rightarrow \frac{\partial T}{\partial r} = 0 \Rightarrow C_5 = 0,$$

$$r = \frac{D}{2} = R \Rightarrow q'' = k\frac{\partial T}{\partial r} \Rightarrow C_4 = \frac{q''}{kD}$$

$$\Rightarrow T(z,r) = \left[\left(\frac{q''}{kD}\right)r^2 + C_6\right]Z(z)$$

$$\int_l^L \int_0^R \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(2r\frac{q''}{kD}\right)\right) - \frac{2V}{\alpha}\left(1 - \left(\frac{r}{R}\right)^2\right)\left[\left(\frac{q''}{kD}\right)r^2 + C_6\right]\frac{\partial T}{\partial z}\right] dr\,dz = 0$$

مساله ۴-۲)

a) $T_{out,tube} < T_{out,diffuser}$

برای این ادعا چند دلیل فیزیکی داریم:

سطح دیفیوزر از سطح لوله بزرگ‌تر است بنابراین شار حرارتی بیشتری به سیال درون دیفیوزر می‌رسد.

از آنجایی‌که $D_2 > D_1$ در سطح مقطع دوم سرعت کمتر از سرعت در سطح مقطع اول است، و این به معنی زمان اقامت بزرگ‌تر و بنابراین افزایش بیشتر دماست .

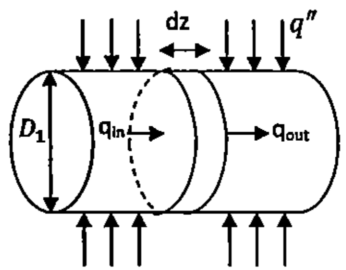
$$q'' A_{tube} = \dot{m} C_p (T_L - T_o), \quad A_{tube} = \pi D_1 L$$

$$q'' A_{diff} = \dot{m} C_p (T'_L - T_o), \quad A_{diff} = \frac{\pi(D_1 + D_2)}{2}L$$

$$\frac{T_L - T_o}{A_{tube}} = \frac{T'_L - T_o}{A_{diff}}, A_{tube} < A_{diff} \Rightarrow T_L - T_o < T'_L - T_o \Rightarrow T_L < T'_L$$

b) فرمولاسیون

در جهت r فرمولاسیون متمرکز (lumped) و در جهت z فرمولاسیون دیفرانسیلی در نظر می‌گیریم

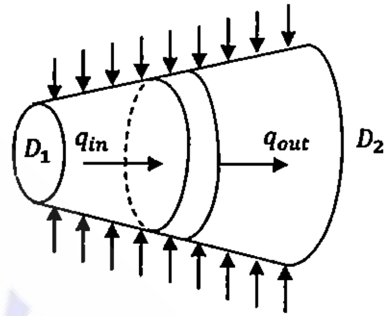


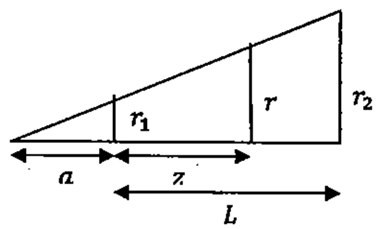
$$q_{in} - q_{out} = 0 \Rightarrow \rho V_1 C_p AT|_z - \rho V_1 C_p AT|_{z+dz} + q'' dA = 0$$

$$\rho V_1 C_p \frac{\pi D_1^2}{4}\left(\frac{T_z - T_{z+dz}}{dz}\right) + q''\pi D_1 = 0 \Rightarrow \frac{dT}{dz} = \frac{4q''}{\rho V_1 C_p D_1}$$

$$\Rightarrow T(z) = \frac{4q''}{\rho V_1 C_p D_1} z + C_2 \quad at\ z = 0: T = T_o \Rightarrow C_2 = T_o$$

$$\Rightarrow T(z) = \frac{4q''}{\rho V_1 C_p D_1} z + T_o \Rightarrow T(L) - T_o = \frac{4q''}{\rho V_1 C_p D_1} L \quad \text{(برای لوله)}$$





$$\frac{r_1}{a} = \frac{r_2}{a+L} = \frac{r}{a+z} \Rightarrow a = \frac{r_1 L}{r_2 - r_1} \Rightarrow r = r_1 + \frac{z}{L}(r_2 - r_1)$$

$$A(z) = \pi r(z)^2 \Rightarrow A(z) = \pi\left(r_1 + \frac{z}{L}(r_2 - r_1)\right)^2, P(z) = 2\pi$$

$$\Rightarrow C_p(\rho V_z A_z T'_z - \rho V_{z+dz} A_{z+dz} T'_{z+dz}) + q'' P(z) dz = 0$$

$\rho V_z A_z = cte$ معادله پیوستگی $\Rightarrow \rho V_z A_z = \rho V_{z+dz} A_{z+dz} = \rho V_1 A_1$

$$\frac{dT'_z}{dz} = \frac{2\pi q''}{\rho V_1 C_p A_1}\left(r_1 + \frac{z}{L}(r_2 - r_1)\right)$$

$$\Rightarrow T'(z) = \frac{2q''}{\rho V_1 C_p r_1^2}\left(r_1 z + \frac{z^2}{L}(r_2 - r_1)\right) + C_1$$

$$at\ z = 0: T' = T_o \Rightarrow C_1 = T_o$$

$$\Rightarrow T'(z) - T_o = \frac{2q''}{\rho V_1 C_p r_1^2}\left(r_1 z + \frac{z^2}{L}(r_2 - r_1)\right)$$

$$T'(L) - T_o = \frac{q''}{\rho V_1 C_p r_1^2}(r_2 + r_1) L \quad \text{(برای دیفیوزر)}$$

$$T(L) - T_o = \frac{2q''}{\rho V_1 C_p r_1} L \quad \text{(برای لوله)}$$

$$\Rightarrow \frac{2q''}{\rho V_1 C_p r_1} < \frac{q''}{\rho V_1 C_p r_1^2}(r_2 + r_1) \Rightarrow T(L) - T_o < T'(L) - T_o$$

$$\Rightarrow T(L) < T'(L)$$

مسأله ۵-۲)

موازنه حرارت برای میعان:

$q_{in} - q_{out} =$ تجمع

$$\Rightarrow \rho_l A \frac{dx(t)}{dt} h_{fg} - hA(T^{Sat} - T_\infty) = \rho_l C_l T^{Sat} A \frac{dx(t)}{dt}$$

$$\Rightarrow q'' = h(T^{Sat} - T_\infty) = \rho_l (h_{fg} - C_l T^{Sat}) \frac{dx(t)}{dt}$$

$$\Rightarrow x(t) = \frac{h(T^{Sat} - T_\infty)}{\rho_l (h_{fg} - C_l T^{Sat})} t + x_o$$

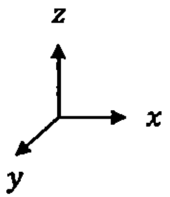
$$\text{At } t = 0 \Rightarrow x(t) = 0 \Rightarrow x_0 = 0 \Rightarrow x(t) = \frac{h(T^{Sat} - T_\infty)}{\rho_l (h_{fg} - C_l T^{Sat})} t$$

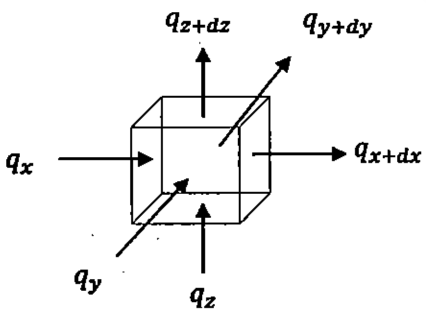
برای نرخ کم میعان شدن (تجمع = صفر) یا برای $h_{fg} > C_1$ خواهیم داشت:

$$x(t) = \frac{h(T^{Sat} - T_\infty)}{\rho_l h_{fg}} t, \Rightarrow q'' = h(T^{Sat} - T_\infty) = \rho_l h_{fg} \frac{dx(t)}{dt} = cte$$

مسأله ۶-۲)

در مختصات کارتزین:





$$q_x - q_{x+dx} + q_y - q_{y+dy} + q_z - q_{z+dz} + \dot{E}_g = \dot{E}_{acc}$$

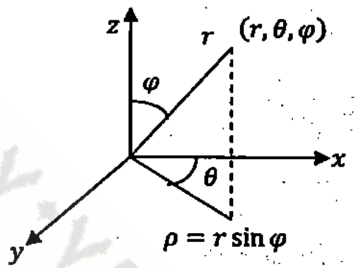
$$\dot{E}_g = \dot{q}(dx\,dy\,dz), \dot{E}_{acc} = \frac{\partial(\rho V C_p T)}{\partial t} = \rho C_p (dx\,dy\,dz) \frac{\partial T}{\partial t}$$

$$q_x = -k(dy\,dz)\frac{\partial T}{\partial x},\ q_y = -k(dx\,dz)\frac{\partial T}{\partial y}, q_z = -k(dy\,dx)\frac{\partial T}{\partial z}$$

$$q = q_x\vec{i} + q_y\vec{j} + q_z\vec{k} = -k\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)T = -k\nabla T$$

$$-\frac{\partial}{\partial x}(q_x)dx-\frac{\partial}{\partial y}(q_y)dy-\frac{\partial}{\partial z}(q_z)dz+\dot{q}(dx\,dy\,dz)=$$
$$\rho C_p(dx\,dy\,dz)\frac{\partial T}{\partial t}$$
$$\frac{\partial}{\partial x}\left(k(dy\,dz)\frac{\partial T}{\partial x}\right)dx+\frac{\partial}{\partial y}\left(k(dx\,dz)\frac{\partial T}{\partial y}\right)dy+\frac{\partial}{\partial z}\left(k(dy\,dx)\frac{\partial T}{\partial z}\right)dz+$$
$$\dot{q}(dx\,dy\,dz)=\rho C_p(dx\,dy\,dz)\frac{\partial T}{\partial t}$$
$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)+\frac{\dot{q}}{k}=\frac{\rho C_p}{k}\frac{\partial T}{\partial t}\Longrightarrow \nabla^2T+\frac{\dot{q}}{k}=\frac{\rho C_p}{k}\frac{\partial T}{\partial t}$$

در مختصات کروی:



برای جلوگیری از بروز اشتباه، حالت زیر را در نظر می‌گیریم:

$$\begin{cases}\text{in } r \text{ direction: } dr \Rightarrow dr\\ \text{in } \varphi \text{ direction: } rd\varphi \Rightarrow dn=rd\varphi\\ \text{in } \theta \text{ direction: } \rho d\theta \Rightarrow dm=\rho d\theta = r\sin\varphi\,d\theta\end{cases}$$

$$q_r-q_{r+dr}+q_n-q_{n+dn}+q_m-q_{m+dm}+\dot{E}_g=\dot{E}_{acc}$$
$$\dot{E}_g=\dot{q}(dr\,dn\,dm),\dot{E}_{acc}=\rho C_p(dr\,dn\,dm)\frac{\partial T}{\partial t}$$
$$q_r=-k(dn\,dm)\frac{\partial T}{\partial r},\ q_n=-k(dr\,dm)\frac{\partial T}{\partial n},q_m=-k(dr\,dn)\frac{\partial T}{\partial m}$$
$$-\frac{\partial}{\partial r}(q_r)dr-\frac{\partial}{\partial n}(q_n)dn-\frac{\partial}{\partial m}(q_m)dm+\dot{q}(dr\,dn\,dm)=$$
$$\rho C_p(dr\,dn\,dm)\frac{\partial T}{\partial t}$$
$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial T}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\theta^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

در مختصات استوانه‌ای:

$$\begin{cases}\text{in } r \text{ direction: } dr \Rightarrow dr\\ \text{in } \varphi \text{ direction: } dz \Rightarrow dz\\ \text{in } \theta \text{ direction: } rd\theta \Rightarrow dn=rd\theta\end{cases}$$

$$q_r-q_{r+dr}+q_z-q_{z+dz}+q_n-q_{n+dn}+\dot{E}_g=\dot{E}_{acc}$$
$$\dot{E}_g=\dot{q}(dr\,dz\,dn),\dot{E}_{acc}=\rho C_p(dr\,dz\,dn)\frac{\partial T}{\partial t}$$
$$q_r=-k(dn\,dz)\frac{\partial T}{\partial r},\ q_z=-k(dr\,dn)\frac{\partial T}{\partial z},q_n=-k(dr\,dz)\frac{\partial T}{\partial n}$$
$$-\frac{\partial}{\partial r}(q_r)dr-\frac{\partial}{\partial z}(q_z)dz-\frac{\partial}{\partial n}(q_n)dn+\dot{q}(dr\,dn\,dz)=$$
$$\rho C_p(dr\,dn\,dz)\frac{\partial T}{\partial t}$$
$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

مساله ۷-۲)

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}+u'''=0\ \Longrightarrow\ \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{u'''}{k}=0$$
$$BC\begin{cases}at\ x=0:\frac{\partial T}{\partial x}=0,at\ x=L:-k\frac{\partial T}{\partial x}=h(T-T_\infty)\\ at\ y=0:\frac{\partial T}{\partial y}=0,at\ y=L:-k\frac{\partial T}{\partial y}=h(T-T_\infty)\end{cases}$$

فرمولاسیون انتگرالی: $\int_0^L\int_0^l\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{u'''}{k}\right)dy\,dx=0$

۱) روش ریتز

$$T-T_\infty=X(x)Y(y),\ \ X(x)=A_1\cos\left(\frac{\pi x}{2L}\right),Y(y)=B_1\cos\left(\frac{\pi y}{2l}\right)$$
$$T-T_\infty=a_0\cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi y}{2l}\right)$$

این شکل از گرادیان دما، شرایط مرزی را ارضا می‌نماید

$$\begin{cases}\frac{\partial T}{\partial x}=-a_0\cos\left(\frac{\pi y}{2l}\right)\cdot\frac{\pi}{2L}\cdot\sin\left(\frac{\pi x}{2L}\right)\ ,\frac{\partial^2 T}{\partial x^2}=-a_0\cos\left(\frac{\pi y}{2l}\right)\cdot\frac{\pi^2}{4L^2}\cdot\cos\left(\frac{\pi x}{2L}\right)\\ \frac{\partial T}{\partial y}=-a_0\cos\left(\frac{\pi x}{2L}\right)\cdot\frac{\pi}{2l}\cdot\sin\left(\frac{\pi y}{2l}\right)\ ,\frac{\partial^2 T}{\partial y^2}=-a_0\cos\left(\frac{\pi x}{2L}\right)\cdot\frac{\pi^2}{4l^2}\cdot\cos\left(\frac{\pi y}{2l}\right)\end{cases}$$
$$\int_0^L\int_0^l\left[-\frac{a_0\pi^2}{4L^2}\cos\left(\frac{\pi y}{2l}\right)\cos\left(\frac{\pi x}{2L}\right)-\frac{a_0\pi^2}{4l^2}\cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi y}{2l}\right)+\frac{u'''}{k}\right]dydx=0$$
$$\Rightarrow\int_0^L\int_0^l\left[\left(\cos\left(\frac{\pi y}{2l}\right)\cos\left(\frac{\pi x}{2L}\right)\right)\left(-\frac{a_0\pi^2(L^2+l^2)}{4L^2l^2}\right)+\frac{u'''}{k}\right]dydx=0$$
$$\cos\left(\frac{\pi y}{2l}\right)\cos\left(\frac{\pi x}{2L}\right)=\frac{1}{2}\left[\cos\left(\frac{\pi y}{2l}+\frac{\pi x}{2L}\right)+\cos\left(\frac{\pi y}{2l}-\frac{\pi x}{2L}\right)\right]$$

$$\Rightarrow \int_0^L\int_0^l\left[\frac{1}{2}\left(\cos\left(\frac{\pi y}{2l}+\frac{\pi x}{2L}\right)+\cos\left(\frac{\pi y}{2l}-\frac{\pi x}{2L}\right)\right)\left(-\frac{a_0\pi^2(L^2+l^2)}{4L^2l^2}\right)+\frac{u'''}{k}\right]dydx=0$$

$$\frac{1}{2}\left(-\frac{a_0\pi^2(L^2+l^2)}{4L^2l^2}\right)=\beta$$

$$\Rightarrow \int_0^L\int_0^l\left[\beta\left(\cos\left(\frac{\pi y}{2l}+\frac{\pi x}{2L}\right)+\cos\left(\frac{\pi y}{2l}-\frac{\pi x}{2L}\right)\right)+\frac{u'''}{k}\right]dydx=0$$

$$\int_0^L\left[\beta\left[\frac{2l}{\pi}\sin\frac{\pi}{2}\left(\frac{y}{l}+\frac{x}{L}\right)+\frac{2l}{\pi}\sin\frac{\pi}{2}\left(\frac{y}{l}-\frac{x}{L}\right)\right]+\frac{u'''y}{k}\right]_0^l dx=0$$

$$\Rightarrow \int_0^L\left[\frac{2l\beta}{\pi}\left[\sin\frac{\pi}{2}\left(1+\frac{x}{L}\right)+\sin\frac{\pi}{2}\left(1-\frac{x}{L}\right)\right]+\frac{u'''l}{k}\right]dx=0$$

$$\Rightarrow \left[\frac{-2l\beta}{\pi}\cdot\frac{2L}{\pi}\left[\cos\frac{\pi}{2}\left(1+\frac{x}{L}\right)-\cos\frac{\pi}{2}\left(1-\frac{x}{L}\right)\right]+\frac{u'''l}{k}x\right]_0^L=0$$

$$\Rightarrow \frac{-4Ll\beta}{\pi^2}\left[\cos\pi-\cos 0-\cos\frac{\pi}{2}+\cos\frac{\pi}{2}\right]+\frac{u'''lL}{k}=0$$

$$\Rightarrow \frac{8Ll\beta}{\pi^2}+\frac{u'''lL}{k}=0 \Rightarrow \frac{8Ll}{\pi^2}\cdot\frac{-a_0\pi^2(L^2+l^2)}{8L^2l^2}+\frac{u'''lL}{k}=0$$

$$\Rightarrow \frac{a_0(L^2+l^2)}{lL}=\frac{u'''lL}{k} \Rightarrow a_0=\frac{u'''L^2l^2}{k(L^2+l^2)}$$

$$\Rightarrow T-T_\infty=\frac{u'''L^2l^2}{k(L^2+l^2)}\cdot\cos\left(\frac{\pi y}{2l}\right)\cos\left(\frac{\pi x}{2L}\right)$$

۲) روش کانتروویچ

$$T-T_\infty=X(x)\cos\left(\frac{\pi y}{2l}\right)$$

$$\begin{cases}\dfrac{\partial T}{\partial x}=X'(x)\cos\left(\dfrac{\pi y}{2l}\right),\ \dfrac{\partial^2 T}{\partial x^2}=X''(x)\cos\left(\dfrac{\pi y}{2l}\right)\\ \dfrac{\partial T}{\partial y}=\dfrac{-\pi}{2l}X(x)\sin\left(\dfrac{\pi y}{2l}\right),\ \dfrac{\partial^2 T}{\partial y^2}=\dfrac{-\pi^2}{4l^2}X(x)\cos\left(\dfrac{\pi y}{2l}\right)\end{cases}$$

$$\Rightarrow \int_0^L\int_0^l\left[X''(x)\cos\left(\frac{\pi y}{2l}\right)-\frac{-\pi^2}{4l^2}X(x)\cos\left(\frac{\pi y}{2l}\right)+\frac{u'''}{k}\right]dxdy=0$$

$$\Rightarrow \int_0^L\left[\frac{X''(x)2l}{\pi}\sin\left(\frac{\pi y}{2l}\right)-\frac{\pi}{2l}X(x)\sin\left(\frac{\pi y}{2l}\right)+\frac{u'''}{k}y\right]_0^l dx=0$$

$$\Rightarrow \int_0^L\left[\frac{X''(x)2l}{\pi}\sin\left(\frac{\pi}{2}\right)-\frac{\pi}{2l}X(x)\sin\left(\frac{\pi}{2}\right)+\frac{u'''}{k}l\right]dx=0$$

این انتگرال برای هر مقدار اختیاری از l برقرار است بنابراین باید عبارت زیر انتگرال صفر باشد:

$$\frac{X''(x)2l}{\pi}-\frac{\pi}{2l}X(x)+\frac{u'''}{k}l=0 \Rightarrow X''(x)-\frac{\pi^2}{4l^2}X(x)=-\frac{u'''\pi}{2k}$$

$$BC\begin{cases}x=L: T=T_\infty \Rightarrow X(L)=0\\ x=0: \dfrac{\partial T}{\partial x}=0 \Rightarrow X'(0)=0\end{cases}$$

$$\Rightarrow X(x)=A\cosh\left(\frac{\pi x}{2l}\right)+B\sinh\left(\frac{\pi x}{2l}\right)+\frac{2u'''l^2}{k\pi}$$

$$X'(0)=0 \Rightarrow B=0$$

$$X(L)=0 \Rightarrow A=\frac{-2u'''l^2}{k\pi\cosh\left(\frac{\pi x}{2l}\right)}$$

$$\Rightarrow X(x)=\frac{2u'''l^2}{k\pi}\cosh\left(\frac{\pi x}{2l}\right)\left[1-\frac{\cosh\left(\frac{\pi x}{2l}\right)}{\cosh\left(\frac{\pi L}{2l}\right)}\right]$$

مساله ۸-۲)

فرمولاسیون متمرکز:

$$q_{in}-q_{out}=q_{acc} \Rightarrow q''A-hA(T-T_\infty)=\frac{\partial}{\partial t}(\rho ALC_pT) \Rightarrow \rho LC_p\frac{\partial T}{\partial t}=q''-h(T-T_\infty)$$

IC: at $t=0 \Rightarrow T=T_\infty,\ \theta=0$

$$\frac{\partial\theta}{\partial t}+\frac{h}{\rho C_pL}\theta=\frac{q''}{\rho C_pL} \Rightarrow \theta=e^{-\frac{h}{\rho C_pL}t}\left[\int e^{\frac{h}{\rho C_pL}t}\frac{q''}{\rho C_pL}+C\right]$$

$$\Rightarrow \theta=\frac{q''}{h}+Ce^{-\frac{h}{\rho C_pL}t} \Rightarrow at\ t=0 \Rightarrow C=-\frac{q''}{h}$$

$$\Rightarrow T-T_\infty=\frac{q''}{h}\left[1-e^{-\frac{h}{\rho C_pL}t}\right]$$

فرمولاسیون انتگرالی:

$$q_{in}-q_{out}=q_{acc} \Rightarrow -\frac{\partial q_x}{\partial t}=\rho VC_p\frac{\partial T}{\partial t}$$

برای قبل از نفوذ: $t\le t_p$

$$\alpha\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t} \Rightarrow \int_0^s\frac{\partial^2 T}{\partial x^2}dx=\frac{1}{\alpha}\int_0^s\frac{\partial T}{\partial t}dx$$ انتگرال موازنه حرارت (HBI)

$$\Rightarrow \left.\frac{\partial T}{\partial x}\right|_{x=s}-\left.\frac{\partial T}{\partial x}\right|_{x=0}=\frac{1}{\alpha}\int_0^s\frac{\partial T}{\partial t}dx$$

فرض: $T = a_0 + a_1x + a_2x^2 \Rightarrow \frac{\partial T}{\partial x} = a_1 + 2a_2x$

$$BC\begin{cases} at\ x=0:\ -k\frac{\partial T}{\partial x} = q'' \Rightarrow a_1 = -\frac{q''}{k} \\ at\ x=s:\ \frac{\partial T}{\partial x} = 0 \Rightarrow a_2 = \frac{q''}{2ks} \\ at\ x=s:\ T = T_\infty \Rightarrow a_0 = T_\infty + \frac{q''s}{2k} \end{cases}$$

$$\Rightarrow T = T_\infty + \frac{q''s}{2k} - \frac{q''}{k}x + \frac{q''}{2ks}x^2$$

$\Rightarrow$ HBI با استفاده از: $\left.\frac{\partial T}{\partial x}\right|_{x=s} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\int_0^s \frac{\partial T}{\partial t}dx \Rightarrow 0 + \frac{q''}{k} = \frac{1}{\alpha}\int_0^s \frac{\partial T}{\partial t}dx$

برای انتگرال لایبنیتز:

$$\frac{d}{dt}\int_0^s T(t,x)dx = \int_0^s \frac{\partial T}{\partial t}dx + T(t,s)\frac{ds}{dt} - T(x,0)\frac{d0}{dx}$$

$$\Rightarrow \frac{d}{dt}\int_0^s T(t,x)dx = \frac{q''\alpha}{k} + T_\infty\frac{ds}{dt}$$

$$\Rightarrow \frac{d}{dt}\int_0^s \left(T_\infty + \frac{q''s}{2k} - \frac{q''}{k}x + \frac{q''}{2ks}x^2\right)dx = \frac{q''\alpha}{k} + T_\infty\frac{ds}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[T_\infty x + \frac{q''s}{2k}x - \frac{q''}{2k}x^2 + \frac{q''}{6ks}x^3\right]_0^s = \frac{q''\alpha}{k} + T_\infty\frac{ds}{dt}$$

$$\Rightarrow T_\infty\frac{ds}{dt} + \frac{q''}{3k}s\frac{ds}{dt} = \frac{q''\alpha}{k} + T_\infty\frac{ds}{dt} \Rightarrow \frac{q''}{3k}s\frac{ds}{dt} = \frac{q''\alpha}{k}$$

$$\Rightarrow \int_0^s s\,ds = 3\alpha\int_0^t dt \Rightarrow s = \sqrt{6\alpha t}$$

برای بعد از نفوذ: $t > t_p$

$$\int_0^L \frac{\partial^2 T}{\partial x^2}dx = \frac{1}{\alpha}\int_0^L \frac{\partial T}{\partial t}dx \Rightarrow \left.\frac{\partial T}{\partial x}\right|_{x=L} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\int_0^L \frac{\partial T}{\partial t}dx$$

فرض: $T = b_0 + b_1x + b_2x^2 \Rightarrow \frac{\partial T}{\partial x} = b_1 + 2b_2x$

$$BC\begin{cases} at\ x=0:\ -k\frac{\partial T}{\partial x} = q'' \Rightarrow b_1 = -\frac{q''}{k} \\ at\ x=L:\ \frac{\partial T}{\partial x} = -\frac{h}{k}(T-T_\infty) \Rightarrow b_1 + 2b_2L = -\frac{h}{k}(T-T_\infty) \end{cases}$$

$$\Rightarrow -\frac{h}{k}\left(b_0 - \frac{q''L}{k} + b_2L^2 - T_\infty\right) = -\frac{q''}{k} + 2b_2L$$

$$\Rightarrow b_0 = \frac{q''}{k} - \frac{2b_2Lk}{h} + \frac{q''L}{k} - b_2L^2 + T_\infty$$

$$\Rightarrow T = \frac{q''}{k} - \frac{2b_2Lk}{h} + \frac{q''L}{k} - b_2L^2 + T_\infty - \frac{q''}{k}x + b_2x^2$$

$$\Rightarrow -\frac{q''}{k} + 2b_2L + \frac{q''}{k} = \frac{1}{\alpha}\frac{db_2}{dt}\int_0^L\left[-\left(\frac{2Lk}{h} + L^2\right) + x^2\right]dx$$

$$\Rightarrow \frac{dT}{db_2} = -\left(\frac{2Lk}{h} + L^2\right) + x^2,\ \ 2b_2L = \frac{1}{\alpha}\frac{db_2}{dt}\int_0^L\left[-\left(\frac{2Lk}{h} + L^2\right) + x^2\right]dx$$

$$\Rightarrow \left(-\frac{Lk}{h} + \frac{L^2}{3}\right)\frac{db_2}{b_2} = \alpha\,dt \Rightarrow -L^2\left(\frac{k}{Lh} + \frac{1}{3}\right)\frac{db_2}{b_2} = \alpha\,dt,\quad Bi = \frac{hL}{k}$$

$$\Rightarrow -L^2\left(\frac{1}{Bi} + \frac{1}{3}\right)\frac{db_2}{b_2} = \alpha\,dt \Rightarrow -L^2\left(\frac{1}{Bi} + \frac{1}{3}\right)(\ln b_2 + \ln C) = \alpha\,dt$$

At $t = t_{pen}, x = L$: $\quad T = T_\infty$, $x = \sqrt{6\alpha t} \Rightarrow t_{pen} = \frac{L^2}{6\alpha}$

At t_{pen} , $T = T_\infty$: $\quad T_\infty = \frac{q''}{k} - \frac{2b_2Lk}{h} + \frac{q''L}{k} - b_2L^2 + T_\infty - \frac{q''}{k}L + b_2L^2$

$$\Rightarrow b_2|_{t_{pen}} = \frac{q''}{2kL} \Rightarrow \ln C = -\ln\frac{q''}{2kL} - \frac{1}{6\left(\frac{1}{Bi}+\frac{1}{3}\right)}$$

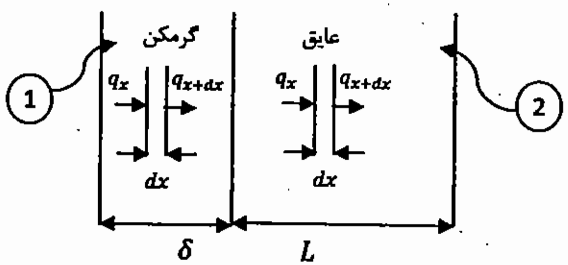
$$\Rightarrow \alpha t = -L^2\left(\frac{1}{Bi} + \frac{1}{3}\right)\ln\frac{2b_2Lk}{q''} + \frac{L^2}{6}$$

$$= -L^2\left(\frac{1}{Bi} + \frac{1}{3}\right)\left(\ln b_2 - \ln\frac{q''}{2kL} - \frac{1}{6\left(\frac{1}{Bi}+\frac{1}{3}\right)}\right) \Rightarrow b_2 = \frac{q''}{2kL}\exp\left(\frac{\frac{1}{6} - \frac{\alpha t}{L^2}}{\frac{1}{Bi}+\frac{1}{3}}\right)$$

$$\Rightarrow T - T_\infty =$$

$$\frac{q''}{kL} - \frac{q''}{2kh}(hL + 2k)\exp\left(\frac{\frac{1}{6} - \frac{\alpha t}{L^2}}{\frac{1}{Bi}+\frac{1}{3}}\right) + \frac{q''}{h} - \frac{q''}{k}x + \frac{q''}{2kL}\exp\left(\frac{\frac{1}{6} - \frac{\alpha t}{L^2}}{\frac{1}{Bi}+\frac{1}{3}}\right)x^2$$

√ **مساله ۹-۲)**



برای گرمکن خواهیم داشت:

$$q_x - q_{x+dx} + u'''dx.A = \rho_1C_1A.dx.\frac{\partial T_1}{\partial t}$$

$$\Rightarrow k_1Adx\frac{\partial^2 T_1}{\partial x^2} + u'''dx.A = \rho_1C_1A.dx.\frac{\partial T_1}{\partial t}$$

$$\Longrightarrow \int_0^\delta \frac{\partial^2 T_1}{\partial x^2}dx + \frac{u'''}{k_1}\int_0^\delta dx = \frac{1}{\alpha_1}\frac{\partial}{\partial t}\int_0^\delta T_1 dx$$

$$BC\begin{cases} at\ x=0:\ \frac{\partial T_1}{\partial x}=0 \\ at\ x=\delta:\ -k\frac{\partial T_1}{\partial x} = -k\frac{\partial T_2}{\partial x} \end{cases}, IC: T_1(x,0)=T_\infty$$

برای عایق خواهیم داشت:

$$q_x - q_{x+dx} = \rho_2 C_2 A.\,dx.\frac{\partial T_2}{\partial t} \Longrightarrow \frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$\Longrightarrow \int_0^\delta \frac{\partial^2 T_2}{\partial x^2}dx = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\int_0^\delta T_2 dx$$

فرض: $\theta_1 = T_1 - T_\infty\ ,\ \theta_2 = T_2 - T_\infty\ ,$

$$\theta_1 = a_0 + a_1 x + a_2 x^2\ ,\ \theta_2 = b_0 + b_1 x + b_2 x^2$$

برای قبل از نفوذ: $t \le t_p$

$$\int_{x=\delta}^{S(t)} \frac{\partial^2 \theta_2}{\partial x^2}dx = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\int_{x=\delta}^{S(t)} \theta_2 dx$$

$$\left.\frac{\partial \theta_2}{\partial x}\right|_{x=S(t)} = 0, \theta_2(S(t),t) = 0, \theta_2(x,0) = 0,$$

$$\theta_2(\delta,t) = \theta_1(\delta,t), k_1\left.\frac{\partial \theta_1}{\partial x}\right|_{x=\delta} = k_2\left.\frac{\partial \theta_2}{\partial x}\right|_{x=\delta}$$

$$\theta_2 = b_0 + b_1 x + b_2 x^2 \Longrightarrow \left.\frac{\partial \theta_2}{\partial x}\right|_{x=S(t)} = b_1 + 2b_2 S(t) = 0$$

$$\Longrightarrow b_1 = -2b_2 S(t), \theta_2(S(t),t) = 0 \Longrightarrow b_0 = b_2 S^2(t)$$

$$\Longrightarrow \theta_2(x,t) = b_2[S^2(t) - 2S(t)x + x^2]$$

$$\int_0^\delta \frac{\partial^2 \theta_1}{\partial x^2}dx + \frac{u'''}{k_1}\delta = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\int_0^\delta \theta_1 dx$$

$$\left.\frac{\partial \theta_1}{\partial x}\right|_{x=0} = 0, \theta_1(\delta,t) = \theta_2(\delta,t), k_1\left.\frac{\partial \theta_1}{\partial x}\right|_{x=\delta} = k_2\left.\frac{\partial \theta_2}{\partial x}\right|_{x=\delta}$$

$$\theta_1 = a_0 + a_1 x + a_2 x^2,\ \left.\frac{\partial \theta_1}{\partial x}\right|_{x=0} = 0 \Longrightarrow a_1 = 0$$

$$k_1\left.\frac{\partial \theta_1}{\partial x}\right|_{x=\delta} = k_2\left.\frac{\partial \theta_2}{\partial x}\right|_{x=\delta} \Longrightarrow a_2 = \frac{k_2 b_2}{k_1 \delta}[\delta - S(t)]$$

$$\theta_1(\delta,t) = \theta_2(\delta,t) \Longrightarrow a_0 = b_2[S^2(t) - 2S(t)\delta + \delta^2] - \frac{k_2 b_2}{k_1}\delta[\delta - S(t)]$$

حال باید با استفاده از معادلات حاکم $S(t)$ و $b_2(t)$ را به دست آوریم:

فرض: $b_2(t) = \frac{A}{S(t)} + B \quad at\ t=0: \theta_2 = 0 \Longrightarrow B = 0$

$$\Longrightarrow b_2(t) = \frac{A}{S(t)}, A = cte \Longrightarrow \theta_2(x,t) = AS(t)\left[1 - 2\left(\frac{x}{S(t)}\right) + \left(\frac{x}{S(t)}\right)^2\right]$$

$$\left.\frac{\partial \theta_2}{\partial x}\right|_{x=S(t)} - \left.\frac{\partial \theta_2}{\partial x}\right|_{x=\delta} = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\int_\delta^{S(t)} \theta_2 dx$$

$$\Longrightarrow -\left(\frac{A}{S(t)}\right)(-2S(t) + 2\delta) = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\left[AS(t)\left(x - \frac{x^2}{S(t)} + \frac{x^3}{3S^2(t)}\right)\right]_\delta^{S(t)}$$

$$\Longrightarrow \left(\frac{2}{S(t)}\right)(S(t) - \delta) = \frac{1}{\alpha_2}\frac{\partial}{\partial t}\left[S(t)\left(\frac{S(t)}{3} - \delta - \frac{\delta^2}{S(t)} + \frac{\delta^3}{3S^2(t)}\right)\right]_\delta^{S(t)}$$

$$\cong \frac{1}{\alpha_2}\frac{\partial}{\partial t}\left[\frac{S^2(t)}{3} - \delta S(t)\right]$$

زیرا داریم: $\delta \ll: \delta^2 \longrightarrow 0\ ,\ \delta^3 \longrightarrow 0$

$$\Longrightarrow \frac{2}{S(t)}[S(t) - \delta] = \frac{1}{\alpha_2}\left[\frac{2}{3}S(t) - \delta\right]\frac{\partial S(t)}{\partial t}$$

$$for\ S(t) \gg \delta: 3\alpha_2 dt = S(t)dS(t) \Longrightarrow 3\alpha_2 t = \frac{S^2(t)}{2} \Longrightarrow S(t) = \sqrt{6\alpha_2 t}$$

اگر $\delta \longrightarrow 0$ آن‌گاه هیچ پروفایل دمایی درون گرمکن وجود نخواهد داشت و در $x = \delta$ شار خروجی از گرمکن به صورت زیر خواهد بود: $q''' = u'''\delta = -k_2\left.\frac{\partial \theta_2}{\partial x}\right|_{x=\delta}$

$$\Longrightarrow \frac{-u'''\delta}{k_2} = AS(t)\left[\frac{-2}{S(t)} + \frac{2\delta}{S^2(t)}\right] = 2A\left[\frac{\delta}{S(t)} - 1\right] = -2A, \delta \ll S(t)$$

$$\Longrightarrow A = \frac{u'''\delta}{2k_2} \Longrightarrow \theta_2(x,t) = \frac{u'''\delta}{2k_2}\left[1 - 2\left(\frac{x}{S(t)}\right) + \left(\frac{x}{S(t)}\right)^2\right], S(t) = \sqrt{6\alpha_2 t}$$

$$\Longrightarrow S(t_p) = L = \sqrt{6\alpha_2 t_p} \Longrightarrow t_p = \frac{L^2}{6\alpha_2}$$

برای بعد از نفوذ: $t > t_p$

فرض می‌کنیم $k_2 \ll$ یعنی یک عایق خوب با رسانایی ضعیف داریم

$$\theta_2(x,t) = b_0 + b_1 x + b_2 x^2, \text{for } k_2 \longrightarrow 0: \left.\frac{\partial \theta_2}{\partial x}\right|_{x=L} = 0 \Longrightarrow b_1 = -2b_2 L$$

$$\theta_2(L,t)=0 \Rightarrow b_0=b_2L^2\,,\ \theta_2(x,t)=b_2L^2\left[1-2\left(\frac{x}{L}\right)+\left(\frac{x}{L}\right)^2\right]$$

و برای θ_1 خواهیم داشت:

$$\theta_1(x,t)=a_0+a_1x+a_2x^2,\ \left.\frac{\partial\theta_1}{\partial x}\right|_{x=0}=0 \Rightarrow a_1=0 \Rightarrow$$

$$\theta_1(x,t)=a_0+a_2x^2$$

$$\text{At } x=\delta: k_1\frac{\partial\theta_1}{\partial x}=k_2\frac{\partial\theta_2}{\partial x}\Rightarrow a_2=\frac{k_2}{k_1\delta}b_2[\delta-L]$$

$$\text{At } x=\delta: \theta_1=\theta_2 \Rightarrow a_0=b_2\left[L^2-2\delta L+\delta^2-\frac{k_2}{k_1}\delta[\delta-L]\right]$$

$$\left.\frac{\partial\theta_1}{\partial x}\right|_{x=\delta}-\left.\frac{\partial\theta_1}{\partial x}\right|_{x=0}+\frac{u'''\delta}{k_1}=\frac{1}{\alpha_1}\frac{\partial}{\partial t}\int_0^\delta\theta_1dx \Rightarrow 2a_2\delta+\frac{u'''\delta}{k_1}=\frac{1}{\alpha_1}\frac{\partial}{\partial t}\int_0^\delta\theta_1dx$$

$$\Rightarrow \frac{2k_2}{k_1}[\delta-L]b_2+\frac{u'''\delta}{k_1}=\frac{1}{\alpha_1}\frac{\partial}{\partial t}\left[a_0x+\frac{a_2}{3}x^3\right]_0^\delta=\frac{1}{\alpha_1}\frac{\partial}{\partial t}\left[a_0\delta+\frac{a_2}{3}\delta^3\right]$$

$$\Rightarrow a_0=Ab_2, a_2=Bb_2, A=L^2-2\delta L+\delta^2-\frac{k_2}{k_1}\delta[\delta-L]=cte$$

$$,B=\frac{k_2}{k_1\delta}[\delta-L]=cte \Rightarrow \frac{2k_2}{k_1}[\delta-L]b_2+\frac{u'''\delta}{k_1}=\frac{1}{\alpha_1}\left[A\delta+\frac{B}{3}\delta^3\right]\frac{\partial b_2}{\partial t}$$

$$\Rightarrow D\frac{\partial b_2}{\partial t}-Eb_2=\frac{u'''\delta}{k_1}\Rightarrow \frac{\partial b_2}{\partial t}-\frac{E}{D}b_2=\frac{u'''\delta}{Dk_1}$$

$$\Rightarrow b_2=e^{\left(\frac{E}{D}\right)t}\left[-\frac{u'''\delta}{k_1E}e^{-\left(\frac{E}{D}\right)t}+F\right]=-\frac{u'''\delta}{k_1E}+Fe^{\left(\frac{E}{D}\right)t}$$

$$\text{At } t=0: \theta_2=0, b_2=0 \Rightarrow F=\frac{u'''\delta}{k_1E}\Rightarrow b_2=\frac{u'''\delta}{k_1E}\left[e^{\left(\frac{E}{D}\right)t}-1\right]$$

مسئله ۱۰-۲)

$$q_{x+dx}-q_x=\rho CA\frac{\partial T}{\partial x}\Rightarrow \frac{\partial^2T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

برای قبل از نفوذ: $t\le t_p$

$$\theta=T-T_0 \Rightarrow \frac{\partial^2\theta}{\partial x^2}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}\Rightarrow \int_0^{S(t)}\frac{\partial^2\theta}{\partial x^2}dx=\frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^{S(t)}\theta dx$$

$$\Rightarrow \left.\frac{\partial\theta}{\partial x}\right|_{x=S(t)}-\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=\frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^{S(t)}\theta dx \quad (*)$$

$$IC: at\ t=0: \theta(x,0)=0,$$

$$BC\begin{cases} at\ x=S(t): \theta(S(t),t)=0 \text{ و } \left.\frac{\partial\theta}{\partial x}\right|_{x=S(t)}=0 \\ at\ x=0: k\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=h(T(0,t)-T_\infty)=h\theta(0,t)+h(T_0-T_\infty)\end{cases}$$

$$\theta(x,t)=b_0+b_1x+b_2x^2$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=S(t)}=0 \Rightarrow b_1=-2b_2S(t)$$

$$\theta(S(t),t)=0 \Rightarrow b_0=b_2S^2(t)$$

$$k\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=h\theta(0,t)+h(T_0-T_\infty)\Rightarrow kb_1=hb_0+h(T_0-T_\infty)$$

$$\Rightarrow -k2b_2S(t)=hb_2S^2(t)+h(T_0-T_\infty)\Rightarrow b_2=\frac{h(T_0-T_\infty)}{S^2(t)\left[h+\frac{2k}{S(t)}\right]}$$

$$\Rightarrow \theta(x,t)=\frac{h(T_0-T_\infty)}{\left[h+\frac{2k}{S(t)}\right]}\left[1-2\left(\frac{x}{S(t)}\right)+\left(\frac{x}{S(t)}\right)^2\right]$$

با جایگزینی این معادله درون (*) خواهیم داشت:

$$-\frac{h}{k}\left[\frac{h}{2k}S(t)-1\right]=\frac{1}{3\alpha}\frac{h}{2k}\frac{\partial S(t)}{\partial t}\Rightarrow \frac{\partial S(t)}{\partial t}+3\alpha\frac{h}{k}S(t)=6\alpha$$

$$\Rightarrow S(t)=e^{\left(-3\alpha\frac{h}{k}\right)t}\left[\frac{6\alpha k}{3\alpha h}e^{\left(3\alpha\frac{h}{k}\right)t}+C\right]$$

$$\text{At } t=0: \theta(x,0)=0 \Rightarrow S(0)=0 \Rightarrow C=-\frac{2k}{h}$$

$$\Rightarrow S(t)=\frac{2k}{h}\left[1-e^{\left(-3\alpha\frac{h}{k}\right)t}\right]$$

برای بعد از نفوذ: $t>t_p$

$$IC: at\ t=t_p, x=0: \theta(0,t_p)=T_0-T_\infty,$$

$$BC\begin{cases} at\ x=0: \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0 \\ at\ x=L: -k\left.\frac{\partial\theta}{\partial x}\right|_{x=L}=h\theta(L,t)\end{cases}$$

$$\theta(x,t)=a_0+a_1x+a_2x^2, \frac{\partial\theta}{\partial x}=a_1+2a_2x\,,$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0 \Rightarrow a_1=0$$

$$-k\frac{\partial\theta}{\partial x}\Big|_{x=L} = h\theta(L,t) \Longrightarrow a_0 = -a_2\left[2\frac{kL}{h}+L\right]$$

$$\Longrightarrow \theta(x,t) = a_2\left[x^2 - L\left(2\frac{kL}{h}+L\right)\right],\ \int_0^L\frac{\partial^2\theta}{\partial x^2}dx = \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^L\theta dx$$

$$\Longrightarrow \frac{\partial\theta}{\partial x}\Big|_{x=L} - \frac{\partial\theta}{\partial x}\Big|_{x=0} = \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^L\theta dx$$

$$\int_0^L\theta dx = a_2\left[\frac{x^3}{3} - L\left(2\frac{kL}{h}+L\right)x\right]_0^L = -2a_2\left[\frac{L^3}{3}+\frac{kL^2}{h}\right]$$

$$\Longrightarrow 2a_2L = \frac{1}{\alpha}\frac{\partial}{\partial t}\left[-2a_2\left[\frac{L^3}{3}+\frac{kL^2}{h}\right]\right] \Longrightarrow a_2\alpha = -L\left[\frac{L}{3}+\frac{k}{h}\right]\frac{\partial a_2}{\partial t}$$

$$\Longrightarrow \frac{-\alpha dt}{L\left[\frac{L}{3}+\frac{k}{h}\right]} = \frac{da_2}{a_2} \Longrightarrow \ln a_2 = \ln m - \frac{\alpha}{L\left[\frac{L}{3}+\frac{k}{h}\right]}t \Longrightarrow a_2 = m.\exp\left(-\frac{\alpha}{L\left[\frac{L}{3}+\frac{k}{h}\right]}t\right)$$

$$\Longrightarrow \theta(x,t) = m.\exp\left(-\frac{\alpha}{L\left[\frac{L}{3}+\frac{k}{h}\right]}t\right)\left[x^2 - L\left(2\frac{k}{h}+L\right)\right]$$

$$\text{At } x=0, t=t_p: S(t_p) = L = \frac{2k}{h}\left[1-e^{\left(-3\alpha\frac{h}{k}\right)t_p}\right]$$

$$\Longrightarrow t_p = -\frac{k}{3\alpha h}\ln\left(1-\frac{Lh}{2k}\right),\ \theta(0,t_p) = T_0 - T_\infty$$

$$\Longrightarrow T_0 - T_\infty = m.\exp\left(-\frac{\alpha}{L\left[\frac{L}{3}+\frac{k}{h}\right]}\cdot\frac{-k}{3\alpha h}\right)\left(1-\frac{Lh}{2k}\right)\left[L\left(2\frac{kL}{h}+L\right)\right]$$

$$\Longrightarrow m = \frac{-(T_0-T_\infty)}{L\left(2\frac{k}{h}+L\right)\left(1-\frac{Lh}{2k}\right)\exp\left(\frac{k/h}{3L\left(\frac{L}{3}+\frac{k}{h}\right)}\right)}$$

$$\text{If } k \gg h: m = -\frac{(T_0-T_\infty)}{\frac{2kL}{h}\exp\left(\frac{1}{3L}\right)}$$

مساله ۱۱-۲)

برای هر المان سوختی استوانه‌ای:

$$q_r - q_{r+dr} + u_0'''\left(1-\left(\frac{r}{R}\right)^2\right)2\pi r dr L = \frac{\partial U}{\partial t}$$

$$= \rho C 2\pi r dr L\frac{\partial T}{\partial t}, \theta = T - T_\infty$$

$$\Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{u_0'''}{k}\left(1-\left(\frac{r}{R}\right)^2\right) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$IC: at\ t=0: \theta = 0$$

$$BC\begin{cases} at\ r=0: \frac{\partial\theta}{\partial r} = 0 \\ at\ r=R: -k\frac{\partial\theta}{\partial r} = h(\theta)\end{cases}$$

h بزرگ است بنابراین مقاومتی درون سیال وجود ندارد و $\theta(R,t)=0$ ، $T(r=R)=T_\infty$

$$\theta(r,t) = a_0 + a_1r + a_2r^2 \Longrightarrow \frac{\partial\theta}{\partial r}\Big|_{r=0} = 0 \Longrightarrow a_1 = 0$$

$$\theta(R,t) = 0 \Longrightarrow a_0 + a_2R^2 = 0 \Longrightarrow a_0 = -a_2R^2$$

$$\Longrightarrow \theta(r,t) = a_2(r^2-R^2) = -a_2R^2\left(1-\left(\frac{r}{R}\right)^2\right), \frac{\partial\theta}{\partial r} = 2a_2r$$

$$\Longrightarrow \int_0^R\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)dr + \frac{u_0'''}{k}\int_0^R\left(1-\left(\frac{r}{R}\right)^2\right)dr = \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^R\theta dr$$

$$\Longrightarrow 4a_2R + \frac{u_0'''}{k}\cdot\frac{2}{3}R = \frac{1}{\alpha}(-R^2)\frac{\partial}{\partial t}\left(a_2\left[r-\frac{r^3}{3R^2}\right]_0^R\right)$$

$$\Longrightarrow 4a_2R + \frac{u_0'''}{k}\cdot\frac{2}{3}R = \frac{-R^2}{\alpha}\cdot\frac{2}{3}R\frac{\partial a_2}{\partial t} \Longrightarrow \frac{\partial a_2}{\partial t} + \frac{6\alpha}{R^2}a_2 = -\frac{u_0'''\alpha}{kR^2}$$

$$\Longrightarrow a_2 = exp\left(\frac{-6\alpha}{R^2}t\right)\left[\int\frac{-u_0'''\alpha}{kR^2}\exp\left(\frac{6\alpha}{R^2}t\right)dt + C\right]$$

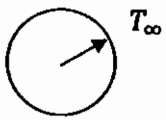
$$at\ t=0: \theta = 0 \Longrightarrow a_2 = 0 \Longrightarrow C = \frac{u_0'''}{6k}$$

$$\Longrightarrow a_2 = -\frac{u_0'''}{6k}\left[1-\exp\left(\frac{-6\alpha}{R^2}t\right)\right]$$

$$\Longrightarrow \theta(r,t) = a_2(r^2-R^2) = \frac{u_0'''}{6k}R^2\left(1-\left(\frac{r}{R}\right)^2\right)\left[1-\exp\left(\frac{-6\alpha}{R^2}t\right)\right]$$

مساله ۱۲-۲)

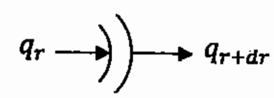
فرمولاسیون متمرکز:



$$h4\pi R^2(T_\infty - T) = \rho C\frac{4}{3}\pi R^3\frac{dT}{dt}, \theta = T - T_\infty \Longrightarrow -h\theta = \frac{\rho CR}{3}\frac{d\theta}{dt}$$

$$\Longrightarrow \theta = A\exp\left(-\frac{3h}{\rho CR}t\right)\ at\ t=0 \Longrightarrow \theta = \theta_0 = T_0 - T_\infty = A$$

$$\Longrightarrow \frac{\theta}{\theta_0} = \frac{T-T_\infty}{T_0-T_\infty} = \exp\left(-\frac{3h}{\rho CR}t\right)$$



فرمولاسیون دیفرانسیلی:

$$q_r - q_{r+dr} = \rho C 4\pi r^2 dr \frac{dT}{dt} \Longrightarrow k\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dr = \rho C r^2 dr \frac{dT}{dt}$$

$$\Longrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \theta = T - T_\infty \Longrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

فرمولاسیون انتگرالی:

برای قبل از نفوذ: $t \le t_p$

$$\int_0^{S(t)} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)dr = \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^{S(t)} \theta dr$$

At $r = S(t)$: $\theta = 0, \frac{\partial \theta}{\partial r} = 0$

At $r = 0$: $-k(4\pi r^2)\frac{\partial \theta}{\partial r} = h(4\pi r^2)[(T_\infty - T_0) - \theta]$

$$\theta(r,t) = a_0 + a_1 r + a_2 r^2 \Longrightarrow \left.\frac{\partial \theta}{\partial r}\right|_{r=S(t)} = 0 \Longrightarrow a_1 = -2a_2 S(t)$$

$$\theta(S(t),t) = a_0 + a_1 S(t) + a_2 S(t)^2 = 0 \Longrightarrow a_0 = a_2 S(t)^2$$

$$\Longrightarrow \theta(r,t) = a_2 S(t)^2\left[1 - 2\frac{r}{S(t)} + \left(\frac{r}{S(t)}\right)^2\right]$$

At $r = 0$: $-k(4\pi r^2)\frac{\partial \theta}{\partial r} = h(4\pi r^2)[(T_\infty - T_0) - \theta]$

$$\Longrightarrow -ka_1 = h[T_\infty - T_0 - a_0] \Longrightarrow -k(-2a_2 S(t)) = h[T_\infty - T_0 - a_2 S(t)^2]$$

$$\Longrightarrow a_2 = \frac{h(T_\infty - T_0)}{S(t)^2\left[h + \frac{2k}{S(t)}\right]} \Longrightarrow \theta(r,t) = \frac{h(T_\infty - T_0)}{\left[h + \frac{2k}{S(t)}\right]}\left[1 - 2\frac{r}{S(t)} + \left(\frac{r}{S(t)}\right)^2\right]$$

برای بعد از نفوذ: $t > t_p$

$$\theta = T - T_\infty \Longrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$IC: at\ t = t_p, r = 0: \theta(0, t_p) = T_0 - T_\infty,$

$$BC\begin{cases} at\ r = 0: \left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0 \\ at\ r = R: -k\left.\frac{\partial \theta}{\partial r}\right|_{r=R} = h\theta(R,t) \end{cases} \qquad \theta(r,t) = b_0 + b_1 r + b_2 r^2$$

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0 \Longrightarrow b_1 = 0 \Longrightarrow \theta(r,t) = b_0 + b_2 r^2$$

$$-k\left.\frac{\partial \theta}{\partial r}\right|_{r=R} = h\theta(R,t) \Longrightarrow -2kb_2 R = h(b_0 + b_2 R^2)$$

$$\Longrightarrow b_0 = -b_2\left(\frac{2kR}{h} + R^2\right) \Longrightarrow \theta(r,t) = b_2\left[r^2 - \left(\frac{2kR}{h} + R^2\right)\right]$$

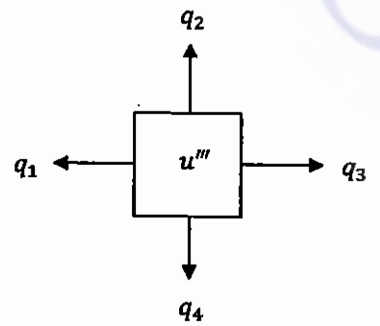
$$\int_0^R \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)dr = \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^R \theta dr \Longrightarrow 6b_2 R\alpha = -2\left[\frac{R^3}{3} + \frac{kR^2}{h}\right]\frac{\partial b_2}{\partial t}$$

$$\Longrightarrow b_2 = B\,exp\left(\frac{-3R\alpha}{\frac{R^3}{3} + \frac{kR^2}{h}}t\right) \Longrightarrow \theta = B\left[r^2 - \left(\frac{2kR}{h} + R^2\right)\right]exp\left(\frac{-3R\alpha}{\frac{R^3}{3} + \frac{kR^2}{h}}t\right)$$

$$IC: at\ t = t_p, r = 0: \theta(0,t_p) = T_0 - T_\infty = -B\left(\frac{2kR}{h} + R^2\right)exp\left(\frac{-3R\alpha}{\frac{R^3}{3} + \frac{kR^2}{h}}t\right)$$

$$\Longrightarrow B = \frac{T_\infty - T_0}{\left(\frac{2kR}{h} + R^2\right)exp\left(\frac{-3R\alpha}{\frac{R^3}{3} + \frac{kR^2}{h}}t\right)}$$

مساله ۱۳-۲)



$$q_1 = q_3, q_2 = q_4$$

$$-(2q_1 A_1 + 2q_2 A_2) + u''' V = \rho C V\frac{dT}{dt}, q = h(T - T_\infty)$$

$$\Longrightarrow -h(T - T_\infty)\left(\frac{l+L}{lL}\right) + u''' = \rho C\frac{dT}{dt}$$

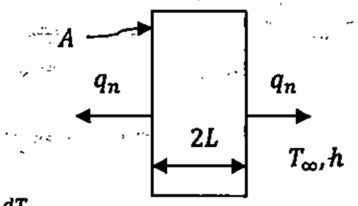
$$\Longrightarrow \frac{dT}{dt} + \frac{h\left(\frac{l+L}{lL}\right)(T - T_\infty)}{\rho C} = \frac{u'''}{\rho C}, T(t=0) = T_\infty$$

$$\theta = T - T_\infty \Longrightarrow \frac{d\theta}{dt} + m\theta = \frac{u'''}{\rho C} \Longrightarrow \theta = \frac{u'''}{\rho C m}(1 - e^{-mt})$$

$\Longrightarrow T - T_\infty = \frac{u'''}{\rho C m}(1 - e^{-mt})$، اگر $h \longrightarrow \infty \Longrightarrow T = \frac{u'''}{\rho C m} + T_\infty$

مسأله ۱۴-۲)

برای مثال ۲-۲ داریم:



$$\Delta S_{gen}^t = \Delta S_{system} + \Delta S_{surround}$$

$$\Delta S_{sur} = \frac{Q}{T_{sur}} = \frac{2q_n}{T_\infty} = \frac{2hA(T-T_\infty)}{T_\infty}$$

$$Q_{sys} = \text{تجمع} = \frac{dE}{dt} = \frac{d(mC_pT)}{dt} = \rho(2AL)C_p\frac{dT}{dt}$$

$$\frac{dS_{sys}}{dt} = \frac{Q_{sys}}{T} = \frac{\rho(2AL)C}{T}\frac{dT}{dt} \Longrightarrow dS_{sys} = \rho(2AL)C_p\frac{dT}{T}$$

$$\Longrightarrow \Delta S_{sys} = \rho(2AL)C_p \ln\frac{T(t)}{T_\infty}$$

$$\Longrightarrow \frac{dS_{tot}}{dt} = \frac{2hA(T-T_\infty)}{T_\infty} + \rho(2AL)C_p\ln\frac{T(t)}{T_\infty}$$

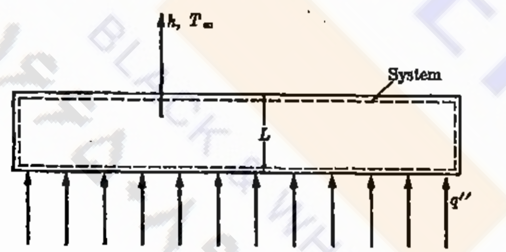
$$\Longrightarrow \Delta S_{gen}^t = \frac{2hA}{T_\infty}\int_0^t (T-T_\infty)dt + \rho(2AL)C_p\int_0^t \ln\left(\frac{T(t)}{T_\infty}\right)dt$$

$$,T - T_\infty = \frac{u'''L}{h}(1-e^{-mt}), m = \frac{h}{\rho C_p L}$$

$$\Longrightarrow \Delta S_{gen}^t = \frac{2hA}{T_\infty}\int_0^t \frac{u'''L}{h}(1-e^{-mt})dt +$$

$$\rho(2AL)C_p\int_0^t \ln\left(1 + \frac{u'''L}{h}(1-e^{-mt})\right)dt$$

برای مثال ۳-۲ داریم:



$$\Delta S_{sur} = \frac{hA(T-T_\infty)}{T_\infty} - \frac{q''A}{T_\infty}$$

$$\frac{dS_{sys}}{dt} = \frac{Q_{sys}}{T} = \frac{\rho(AL)C}{T}\frac{dT}{dt} \Longrightarrow dS_{sys} = \rho(AL)C_p\frac{dT}{T}$$

$$\Longrightarrow \Delta S_{sys} = \rho(AL)C_p\ln\frac{T(t)}{T_\infty}$$

$$\Longrightarrow \frac{dS_{tot}}{dt} = \frac{hA(T-T_\infty)}{T_\infty} - \frac{q''A}{T_\infty} + \rho(AL)C_p\ln\frac{T(t)}{T_\infty}$$

$$\Longrightarrow \Delta S_{gen}^t = \frac{1}{T_\infty}\int_0^t [hA(T-T_\infty) - q''A]dt + \rho(AL)C_p\int_0^t \ln\left(\frac{T(t)}{T_\infty}\right)dt$$

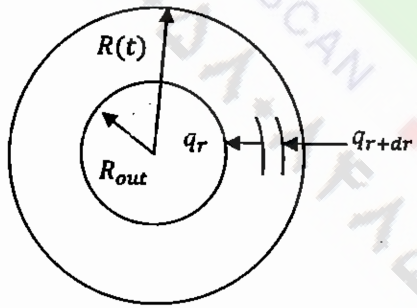
$$,T - T_\infty = \frac{q''}{h}(1-e^{-mt}), m = \frac{h}{\rho C_p L}$$

$$\Longrightarrow \Delta S_{gen}^t = \frac{1}{T_\infty}\int_0^t \left[hA\frac{q''}{h}(1-e^{-mt}) - q''A\right]dt$$

$$+\rho(AL)C_p\int_0^t \ln\left(1 + \frac{u'''L}{h}(1-e^{-mt})\right)dt$$

مسأله ۱۵-۲)

برای بخش جامد خواهیم داشت:



$$q_{r+dr} - q_r = \rho_s C_s(2\pi r dr L)\frac{\partial T}{\partial t}$$

$$\Longrightarrow k_1 2\pi L\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \rho_s C_s(2\pi r L)\frac{\partial T}{\partial t}$$

$$\Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha_s}\frac{\partial \theta}{\partial t}, \theta = T - T_g$$

$$IC: at\ t = 0: \theta(r,0) = T_f - T_g$$

$$BC\begin{cases}\theta(R(t),t) = T_f - T_g \\ k_s 2\pi R(t) L\left.\frac{\partial \theta}{\partial r}\right|_{R(t)} = h_{fs}\frac{\partial m_s}{\partial t} = h_{fs}\rho_s\frac{\partial V_s(t)}{\partial t}\end{cases}$$

$$dV_s(t) = 2\pi R(t) L\rho_s dR(t) \Longrightarrow k_s\left.\frac{\partial \theta}{\partial r}\right|_{R(t)} = h_{fs}\rho_s\frac{\partial R(t)}{\partial t}$$

با صرف‌نظر از ضخامت لوله:

$$at\ r = R_{out}: k_s(2\pi R_{out}L)\left.\frac{\partial \theta}{\partial r}\right|_{R_{out}} = \frac{T_f - T_g}{\frac{\ln\left(\frac{R_{in}}{R_{out}}\right)}{2k_s\pi L} + \frac{1}{h_\infty(2\pi R_{in})L} + \frac{\ln\left(\frac{R_{out}}{R_{in}}\right)}{2k^*\pi L}}$$

$$\Longrightarrow \left.\frac{\partial \theta}{\partial r}\right|_{R_{out}} = \frac{(T_f - T_g)/R_{out}}{\ln(R(t)) + \left[\frac{k_s}{h_\infty \cdot R_{in}} + \frac{k_s \cdot \ln\left(\frac{R_{out}}{R_{in}}\right)}{k^*} - \ln(R_{out})\right]} = \frac{(T_f - T_g)/R_{out}}{\ln(R(t)) + A}$$

فرض می‌کنیم که نرخ یخ‌زدن کوچک باشد بنابراین: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)=0$

$$\int_{R_{out}}^{R(t)}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)dr=0\Rightarrow R(t)\frac{h_{fs}\rho_s}{k_s}\cdot\frac{dR(t)}{dt}=\frac{(T_f-T_g)}{\ln(R(t))+A}$$

$$\Rightarrow\int R(t)[\ln(R(t))+A]dR(t)=\frac{k_s}{h_{fs}\rho_s}\int(T_f-T_g)dt$$

$$\Rightarrow\frac{R^2(t)}{2}\left[\ln(R(t))+A-\frac{1}{2}\right]=\frac{k_s(T_f-T_g)}{h_{fs}\rho_s}t+B$$

$$at\ t=0, R(t)=R_{out}\Rightarrow B=\frac{R_{out}^2}{2}\left[\ln(R_{out})+A-\frac{1}{2}\right]$$

$$A=k_s\left(\frac{1}{h_\infty R_{in}}+\frac{\ln\left(\frac{R_{out}}{R_{in}}\right)}{k^*}\right)-\ln(R_{out})$$

مساله ۱۶-۲)

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$$

$$IC: T(x,0)=T_s,\ \ BC\begin{cases}k\frac{\partial T(0,t)}{\partial x}=h(T(0,t)-T_\infty)\\ T(X(t),t)=T_s\\ k\frac{\partial T(X(t),t)}{\partial x}=\rho h_{vc}\frac{dX}{dt}\end{cases}$$

$$\theta=T-T_\infty\Rightarrow\frac{\partial\theta}{\partial t}=\alpha\frac{\partial^2\theta}{\partial x^2}$$

$$\Rightarrow IC:\theta(x,0)=\theta_s,\ BC\begin{cases}k\frac{\partial\theta(0,t)}{\partial x}=h\theta(0,t)\ (1)\\ \theta(X,t)=\theta_s\quad(2)\\ k\frac{\partial\theta(X,t)}{\partial x}=\rho h_{vc}\frac{dX}{dt}\quad(3)\end{cases}$$

a) فرمولاسیون انتگرالی با دیدگاه فیزیکی:

$$\frac{d}{dt}\int_0^X\rho CT\,dx=-q_x=k\left(\frac{\partial T}{\partial x}\right)_{x=X(t)}\Rightarrow\frac{1}{\alpha}\frac{d}{dt}\int_0^X T\,dx=\left(\frac{\partial T}{\partial x}\right)_{x=X(t)}$$

$$\Rightarrow\frac{1}{\alpha}\frac{d}{dt}\int_0^X\theta\,dx=\left(\frac{\partial\theta}{\partial x}\right)_{x=X(t)}$$

b) حل تقریبی مساله به صورت چند جمله‌ای با استفاده از روش کانتروویچ:

$$\theta(x,t)=a_0+a_1x+a_2x^2$$

$$(1)\Rightarrow a_1=\frac{a_0h}{k}$$

$$(2)\Rightarrow a_0\left(1+\frac{h}{k}X\right)=\theta_s-a_2X^2$$

$$\Rightarrow\theta(x,t)=\frac{\theta_s-a_2X(t)^2}{\left(1+\frac{h}{k}X(t)\right)}+\frac{h}{k}\frac{\theta_s-a_2X(t)^2}{\left(1+\frac{h}{k}X(t)\right)}x+a_2x^2$$

مساله ۱۷-۲)

a) تولید آنتروپی کل: $S_g=\Delta S_{sys}+\Delta S_{sur}$

$$\Delta S_{sur}=\frac{Q}{T_{sur}}=\frac{Q}{T_\infty},Q=q''\cdot A_s, dA_s=2\pi r dx=\pi D dx\ ,D=\left(\frac{D_2-D_1}{L}x+D_1\right)$$

$$A_s=\int_0^L\pi\left(\frac{D_2-D_1}{L}x+D_1\right)dx=\pi\left[\frac{D_2-D_1}{2L}x^2+D_1x\right]_0^L=\pi\left(\frac{D_2+D_1}{2}\right)L$$

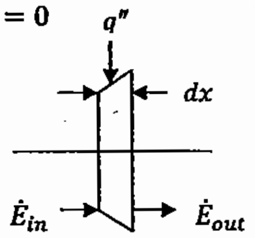
$$\Delta S_{sur}=\frac{q''}{T_\infty}\pi\left(\frac{D_2+D_1}{2}\right)L$$

$$dS_{sys}=\frac{dQ}{T}\Rightarrow\Delta S_{sys}=\int_{x=0}^{x=L}\frac{dQ}{T}\quad(1)$$

$$dQ=q''\cdot dA_s=q''\pi\left(\frac{D_2-D_1}{L}x+D_1\right)dx$$

برای به دست آوردن T خواهیم داشت:

$$\dot{E}_{in}-\dot{E}_{out}=0\Rightarrow\dot{m}C_pT\Big|_x-\dot{m}C_pT\Big|_{x+dx}+q''dA_s=0$$



$$\Rightarrow-\dot{m}C_p\frac{dT}{dx}dx+q''\pi\left(\frac{D_2-D_1}{L}x+D_1\right)dx=0$$

$$\Rightarrow-\rho u_{in}A_{in}C_p\frac{dT}{dx}+q''\pi\left(\frac{D_2-D_1}{L}x+D_1\right)=0$$

$$\Rightarrow\rho u_{in}\frac{\pi}{4}D_1^2C_p\frac{dT}{dx}-q''\pi\left(\frac{D_2-D_1}{L}x+D_1\right)=0$$

$$\Rightarrow\frac{dT}{dx}=\frac{q''\pi\left(\frac{D_2-D_1}{L}x+D_1\right)}{\rho u_{in}\frac{\pi}{4}D_1^2C_p}\Rightarrow\int_0^T dT=\frac{q''\pi}{\rho u_{in}\frac{\pi}{4}D_1^2C_p}\int_0^x\left(\frac{D_2-D_1}{L}x+D_1\right)dx$$

$$\Rightarrow T=\frac{q''\pi}{\rho u_{in}\frac{\pi}{4}D_1^2C_p}\left(\frac{D_2-D_1}{L}x^2+D_1x\right)+C$$

$$BC: at\ x=0\Rightarrow T=T_{in}\Rightarrow C=T_{in}$$

با جایگزینی T و dQ درون معادله (۱)، ΔS_{sys} حاصل خواهد شد.

b) در این بخش توزیع دما تغییر نموده و انتقال حرارت هدایتی را در المان در نظر می‌گیریم

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$\Longrightarrow \dot{m}C_pT\Big|_x - \dot{m}C_pT\Big|_{x+dx} + q_xA_x - q_{x+dx}A_{x+dx} + q''dA_s = 0$$

$$A_x = \pi r^2 = \pi\frac{D^2}{4} = \frac{\pi}{4}\left[\frac{D_2-D_1}{L}x + D_1\right]^2$$

$$\Longrightarrow -\dot{m}C_p\frac{dT}{dx}dx - \frac{d(q_xA_x)}{dx}dx + q''\pi\left(\frac{D_2-D_1}{L}x + D_1\right)dx = 0$$

$$\Longrightarrow -\dot{m}C_p\frac{dT}{dx}dx - k\frac{\pi}{4}\frac{d}{dx}\left(\left[\frac{D_2-D_1}{L}x + D_1\right]^2\frac{dT}{dx}\right)dx + q''\pi\left(\frac{D_2-D_1}{L}x + D_1\right)dx = 0$$

$\Longrightarrow$ با حل این معادله دما به دست خواهد آمد

$$BC\begin{cases} at\ x = 0 \Longrightarrow T = T_{in} \\ at\ x = L \Longrightarrow T = T_{out}\end{cases}$$

مساله ۱۸-۲)

a)

$$q_\xi = -k_\xi\frac{\partial T}{\partial \xi} \text{ and } q_\eta = -k_\eta\frac{\partial T}{\partial \eta}$$

$$q_x = q_\xi\Big|_x - q_\eta\Big|_x = q_\xi\cdot\sin\alpha - q_\eta\cdot\cos\alpha = -k_\xi\frac{\partial T}{\partial \xi}\sin\alpha + k_\eta\frac{\partial T}{\partial \eta}\cos\alpha$$

$$q_y = q_\xi\Big|_y + q_\eta\Big|_y = q_\xi\cdot\cos\alpha + q_\eta\cdot\sin\alpha = -k_\xi\frac{\partial T}{\partial \xi}\cos\alpha + k_\eta\frac{\partial T}{\partial \eta}\sin\alpha$$

b)

$$x = \xi\cos\left(\frac{\pi}{2}-\alpha\right) = \xi\sin\alpha \Longrightarrow \frac{\partial x}{\partial \xi} = \sin\alpha$$

$$y = \xi\sin\left(\frac{\pi}{2}-\alpha\right) = \xi\cos\alpha \Longrightarrow \frac{\partial y}{\partial \xi} = \cos\alpha$$

$$x = -\eta\cos\alpha \Longrightarrow \frac{\partial x}{\partial \eta} = -\cos\alpha$$

$$y = \eta\cos\left(\frac{\pi}{2}-\alpha\right) = \eta\sin\alpha \Longrightarrow \frac{\partial y}{\partial \eta} = \sin\alpha$$

$$\frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \xi} = \frac{\partial T}{\partial x}\cdot\sin\alpha + \frac{\partial T}{\partial y}\cdot\cos\alpha$$

$$\frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \eta} = -\frac{\partial T}{\partial x}\cdot\cos\alpha + \frac{\partial T}{\partial y}\cdot\sin\alpha$$

$$q_x = -k_\xi\frac{\partial T}{\partial \xi}\sin\alpha + k_\eta\frac{\partial T}{\partial \eta}\cos\alpha = (-k_\xi\sin\alpha)\left(\frac{\partial T}{\partial x}\cdot\sin\alpha + \frac{\partial T}{\partial y}\cdot\cos\alpha\right) + (k_\eta\cos\alpha)\left(-\frac{\partial T}{\partial x}\cdot\cos\alpha + \frac{\partial T}{\partial y}\cdot\sin\alpha\right)$$

$$= (-k_\xi\sin^2\alpha)\frac{\partial T}{\partial x} + \left(-k_\xi\sin\alpha\cos\alpha\frac{\partial T}{\partial y}\right) + (-k_\eta\cos^2\alpha)\frac{\partial T}{\partial x} + (k_\eta\cos\alpha\sin\alpha)\frac{\partial T}{\partial y}$$

$$\Longrightarrow q_x = -(k_\xi\sin^2\alpha + k_\eta\cos^2\alpha)\frac{\partial T}{\partial x} - (k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial T}{\partial y}$$

$$q_y = -k_\xi\frac{\partial T}{\partial \xi}\cos\alpha - k_\eta\frac{\partial T}{\partial \eta}\sin\alpha = (-k_\xi\cos\alpha)\left(\frac{\partial T}{\partial x}\cdot\sin\alpha + \frac{\partial T}{\partial y}\cdot\cos\alpha\right) - (k_\eta\sin\alpha)\left(-\frac{\partial T}{\partial x}\cdot\cos\alpha + \frac{\partial T}{\partial y}\cdot\sin\alpha\right)$$

$$= (-k_\xi\cos^2\alpha)\frac{\partial T}{\partial y} + \left(-k_\xi\sin\alpha\cos\alpha\frac{\partial T}{\partial x}\right) + (-k_\eta\cos^2\alpha)\frac{\partial T}{\partial y} + (k_\eta\cos\alpha\sin\alpha)\frac{\partial T}{\partial x}$$

$$\Longrightarrow q_y = -(k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial T}{\partial x} - (k_\xi\cos^2\alpha + k_\eta\sin^2\alpha)\frac{\partial T}{\partial y}$$

c) معادله کلی:

$$\frac{DT}{Dt} = \alpha\nabla^2T + \frac{q''}{\rho c_p}$$

فرض می‌کنیم معادله دو بعدی، بدون تولید حرارت، و بدون حرکت توده است:

$$\Longrightarrow \rho c_p\frac{\partial T}{\partial t} = -\left[\frac{dq_x}{dx} + \frac{dq_y}{dy}\right] = -\left[-(k_\xi\sin^2\alpha + k_\eta\cos^2\alpha)\frac{\partial^2 T}{\partial x^2} - (k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial T}{\partial x\partial y}\right] - \left[-(k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial^2 T}{\partial y\partial x} - (k_\xi\cos^2\alpha + k_\eta\sin^2\alpha)\frac{\partial^2 T}{\partial y^2}\right]$$

$$\Longrightarrow \rho c_p\frac{\partial T}{\partial t} = (k_\xi\sin^2\alpha + k_\eta\cos^2\alpha)\frac{\partial^2 T}{\partial x^2} + (k_\xi\cos^2\alpha + k_\eta\sin^2\alpha)\frac{\partial^2 T}{\partial y^2} + 2(k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial^2 T}{\partial y\partial x}$$

d) معادله هدایتی برای سیستم ایزوتروپیک: $k_\xi = k_\eta = k$

$$\Longrightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

e)

$$q_{ave} = \frac{T_2 - T_1}{\frac{L}{k_{ave}\cdot A}} = \frac{\int_0^L q_y A dy}{\int_0^L dy}, \quad q_y = -k_\xi \frac{\partial T}{\partial \xi}\cos\alpha - k_\eta \frac{\partial T}{\partial \eta}\sin\alpha$$

$$\int_0^L q_y dy = -k_\xi \cos\alpha \int_0^L \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \xi}dy - k_\eta \sin\alpha \int_0^L \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \eta}dy$$

$$= -k_\xi \cos^2\alpha\,(T_1 - T_2) - k_\eta \sin^2\alpha\,(T_1 - T_2)$$

$$\Longrightarrow k_{ave} = k_\xi \cos^2\alpha + k_\eta \sin^2\alpha$$

f)

$$\tan\beta = \frac{q_\eta}{q_\xi} = \frac{k_\eta(\partial T/\partial \eta)}{k_\xi(\partial T/\partial \xi)}$$

$$\frac{\partial T}{\partial \xi} = \frac{\partial T}{\partial x}\cdot\sin\alpha + \frac{\partial T}{\partial y}\cdot\cos\alpha, \quad \frac{\partial T}{\partial \eta} = -\frac{\partial T}{\partial x}\cdot\cos\alpha + \frac{\partial T}{\partial y}\cdot\sin\alpha$$

$$\Longrightarrow \tan\beta = \frac{k_\eta[\sin\alpha(\partial T/\partial y) - \cos\alpha(\partial T/\partial x)]}{k_\xi[\cos\alpha(\partial T/\partial y) - \sin\alpha(\partial T/\partial x)]}$$

$$\text{If}\ \frac{\partial T}{\partial t} = 0 \Longrightarrow \tan\beta = \frac{k_\eta}{k_\xi}\frac{\sin\alpha}{\cos\alpha} = \tan\alpha\cdot\frac{k_\eta}{k_\xi}$$

مساله ۱۹-۲)

سیال=۱ و ضخامت لوله =۲

(i)

برای ۲: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$

برای ۱: $q_{r+dr} - q_r + q_z - q_{z+dz} = \rho_1 c_1 (2\pi r dr) dz \frac{\partial T_1}{\partial t}$

که در آن q_z جریان آنتالپی است و از هدایت محوری صرف‌نظر نموده‌ایم

$$\Longrightarrow k_1 2\pi dz \frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right)dr - \rho_1 c_1 U(2\pi r dr)dz\frac{\partial T_1}{\partial z} = \rho_1 c_1 2\pi r dr\cdot dz\frac{\partial T_1}{\partial t}$$

$$\Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

برای ضخامت زیاد:

برای قبل از نفوذ: $t < t_p$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$IC: T_2(r,0) = T_0,\ BC\begin{cases} at\ r = R_{out}: k_2\frac{\partial T_2}{\partial r} = q'' \\ at\ r = S(t): T_2 = T_0, \frac{\partial T_2}{\partial r} = 0 \end{cases}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: T_1(r,0) = T_0,\ BC\begin{cases} at\ r = R_{in}: T_2 = T_1, k_1\frac{\partial T_1}{\partial r} = k_2\frac{\partial T_2}{\partial r} \\ at\ r = S(t): T_1 = T_0, \frac{\partial T_1}{\partial r} = 0 \\ at\ z = 0: T_1(z = 0, r, t) = T_0 \end{cases}$$

برای بعد از نفوذ: $t > t_p$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$IC: T_2(r,t_p) = T_0,\ BC\begin{cases} at\ r = R_{out}: k_2\frac{\partial T_2}{\partial r} = q'' \\ at\ r = R_{in}: T_2 = T_1, k_1\frac{\partial T_1}{\partial r} = k_2\frac{\partial T_2}{\partial r} \end{cases}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: T_1(r,t_p) = T_0,\ BC\begin{cases} at\ r = R_{in}: T_2 = T_1, k_1\frac{\partial T_1}{\partial r} = k_2\frac{\partial T_2}{\partial r} \\ at\ r = 0: \frac{\partial T_1}{\partial r} = 0 \\ at\ z = 0: T_1(z = 0, r, t) = T_0 \end{cases}$$

(ii) در این شرایط از معادله ضخامت لوله صرفنظر می‌شود و خواهیم داشت:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: T_1(z,r,0) = T_0\ , BC \begin{cases} at\ r = R: q'' = k_1 \dfrac{\partial T_1}{\partial r} \\ at\ r = 0: \dfrac{\partial T_1}{\partial r} = 0 \\ at\ z = 0: T_1(z = 0, r, t) = T_0 \end{cases}$$

(iii) برای ضخامت کم زمان نفوذ در نظر گرفته نمی‌شود: $t_p = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: T_1(z,r,0) = T_0\ , BC \begin{cases} at\ r = R_{in}: T_2 = T_1, k_1 \dfrac{\partial T_1}{\partial r} = k_2 \dfrac{\partial T_2}{\partial r} \\ at\ r = 0: \dfrac{\partial T_1}{\partial r} = 0 \\ at\ z = 0: T_1(z = 0, r, t) = T_0 \end{cases}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$IC: T_2(r,0) = T_0\ , BC \begin{cases} at\ r = R_{out}: k_2 \dfrac{\partial T_2}{\partial r} = q'' \\ at\ r = R_{in}: T_2 = T_1, k_1 \dfrac{\partial T_1}{\partial r} = k_2 \dfrac{\partial T_2}{\partial r} \end{cases}$$

(iv) سیال به صورت محوری متمرکز است بنابراین:

a) ضریب انتقال حرارت بزرگ است: $h \gg$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$IC: T_2(R_{in}, t_p) = T_0\ , BC \begin{cases} at\ r = R_{out}: k_2 \dfrac{\partial T_2}{\partial r} = q'' \\ at\ r = R_{in}: T_2 = T_1(x,t) \end{cases}$$

$$q''(2\pi R_{in} L) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: T_1(x, t_p) = T_0\ , BC: T_1(0,t) = T_0$$

b) ضریب انتقال حرارت متوسط است:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$$

$$IC: T_2(R_{in}, t_p) = T_0\ , BC \begin{cases} at\ r = R_{out}: k_2 \dfrac{\partial T_2}{\partial r} = q'' \\ at\ r = R_{in}: k_2 \dfrac{\partial T_2}{\partial r} = h\big(T_2 - T_1(x,t)\big) \end{cases}$$

(v) سیال و لوله به صورت شعاعی متمرکزند:

برای لوله:

$$q''(2\pi R_{out} L) - h(2\pi R_{in} L)\big(T_2 - T_1(x,t)\big) = \rho_2 c_2 \pi (R_{out}^2 - R_{in}^2)\frac{dT_2}{dt}$$

$$IC: at\ t = 0: T_2 = T_0$$

برای سیال:

$$h(2\pi R_{in} L)\big(T_2 - T_1(x,t)\big) = \frac{1}{\alpha_1}\left[U\frac{\partial T_1}{\partial z} + \frac{\partial T_1}{\partial t}\right]$$

$$IC: at\ t = 0: T_1 = T_0\ , BC: at\ x = 0: T_1 = T_0$$

مساله ۲۰-۲)

در حالت پایا تابعیت زمانی نداریم. از انتقال حرارت هدایتی در جهت x صرف‌نظر می‌کنیم. سیال ایده‌ال است و عبارت اتلاف حرارتی نداریم، بنابراین:

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\left[u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y}\right], u_y = 0\ , u_x = U_\infty$$

$$\theta = T - T_\infty \Rightarrow \frac{\partial^2\theta}{\partial y^2} = \frac{1}{\alpha}\left[U_\infty\frac{\partial\theta}{\partial x}\right], \mathrm{BC}\begin{cases} \dfrac{\partial\theta}{\partial y}(x,\delta(x)) = 0 & (1) \\ \theta(x,\delta(x)) = 0 & (2) \\ \theta(0,y) = \theta_w & (3) \end{cases}$$

$$\theta(x,y) = a_0 + a_1 y + a_2 y^2$$

$$(1) \Rightarrow a_1 = -2a_2\delta$$

$$(2) \Rightarrow a_0 = a_2\delta^2$$

$$\theta(x,y) = a_2\delta^2\left[1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)^2\right]$$

$$\int_0^{\delta(x)}\frac{\partial^2\theta}{\partial y^2}dy = \frac{U_\infty}{\alpha}\frac{\partial}{\partial x}\int_0^{\delta(x)}\theta dy$$

$$\Rightarrow \frac{\partial\theta}{\partial y}\Big|_{y=\delta(x)} - \frac{\partial\theta}{\partial y}\Big|_{y=0} = \frac{U_\infty}{\alpha}\frac{\partial}{\partial x}\int_0^{\delta(x)} a_2\delta^2\left[1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)^2\right]dy$$

$$\Rightarrow 2a_2\delta = \frac{U_\infty}{\alpha}\frac{\partial}{\partial x}\left(a_2\frac{\delta^3}{3}\right), a_2 = f(\delta) = \frac{A}{\delta}$$

$$\Rightarrow U_\infty \int_0^x \frac{\partial}{\partial x}\delta^2 dx = 6\alpha \int_0^x dx$$

$$\delta(x) = A_0 x + A_1$$

$$at\ x = 0: \delta = 0 \Rightarrow \delta(x) = A_0 x \Rightarrow U_\infty (A_0 x)^2 = 6\alpha x$$

$$\Rightarrow \delta(x) = \sqrt{\frac{6\alpha}{xU_\infty}}$$

مسئله (۲۱-۲)

از ضخامت صفحه صرف‌نظر نمی‌کنیم، بنابراین حل شامل دو بخش است:

برای قبل از نفوذ: $t < t_p$

$$q_y - q_{y+dy} = \rho c A dy \frac{\partial T_1}{\partial t} \Rightarrow \frac{\partial^2 T_1}{\partial y^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t}, \theta_1 = T_1 - T_\infty$$

$$\Rightarrow \frac{\partial^2 \theta_1}{\partial y^2} = \frac{1}{\alpha_1}\frac{\partial \theta_1}{\partial t} \qquad IC: at\ t = 0: \theta_1(y,t) = 0,$$

$$BC: \begin{cases} at\ y = 0: -k\frac{\partial \theta_1}{\partial y} = q'' \\ at\ y = S(t) \text{ :عمق نفوذ}: \theta_1 = 0, \frac{\partial \theta_1}{\partial y} = 0 \end{cases}$$

$$\theta_1(y,t) = a_2 + a_1 y + a_2 y^2$$

$$at\ y = 0: -k_1 \frac{\partial \theta_1}{\partial y} = q'' \Rightarrow a_1 = -\frac{q''}{k_1}$$

$$at\ y = S(t): \frac{\partial \theta_1}{\partial y} = 0 \Rightarrow a_1 = -2a_2 S(t)$$

$$at\ y = S(t): \theta_1 = 0 \Rightarrow a_2 = \frac{q''/k_1}{2S(t)}, a_0 = \frac{q''/k_1}{2}S(t)$$

$$\Rightarrow \theta_1(y,t) = \frac{q''}{2k_1}S(t)\left[1 - 2\frac{y}{S(t)} + \left(\frac{y}{S(t)}\right)^2\right]$$

$$\Rightarrow \int_0^{S(t)} \frac{\partial^2 \theta_1}{\partial y^2} dy = \frac{1}{\alpha_1}\frac{\partial}{\partial t}\int_0^{S(t)} \theta_1 dy \Rightarrow \left.\frac{\partial \theta_1}{\partial y}\right|_{y=S(t)} - \left.\frac{\partial \theta_1}{\partial y}\right|_{y=0} =$$

$$\frac{1}{\alpha_1}\frac{\partial}{\partial t}\left(\frac{q''}{2k_1}S(t)\left[y - \frac{y^2}{S(t)} + \frac{y^3}{3S^2(t)}\right]_0^{S(t)}\right) \Rightarrow \alpha_1 \frac{q''}{k_1} = \frac{q''}{2k_1}\frac{\partial}{\partial t}\left[\frac{S^2(t)}{3}\right]$$

$$\Rightarrow \int d\,S^2(t) = \int 6\alpha_1 dt \Rightarrow S^2(t) = 6\alpha_1 t + A, S(t=0) = 0 \Rightarrow A = 0$$

$$\Rightarrow S(t) = \sqrt{6\alpha_1 t}$$

$$\theta_1(y,t) = \frac{q''}{2k_1}\sqrt{6\alpha_1 t}\left[1 - 2\frac{y}{\sqrt{6\alpha_1 t}} + \left(\frac{y}{\sqrt{6\alpha_1 t}}\right)^2\right]$$

$$S(t_p) = \delta^* = \sqrt{6\alpha_1 t_p} \Rightarrow t_p = \frac{(\delta^*)^2}{6\alpha_1}$$

برای بعد از نفوذ:

برای سیال:

$$q_x - q_{x+dx} + q_y - q_{y+dy} = \rho_1 c_1 dx\, dy \frac{\partial T_1}{\partial t}$$

$$q_x = \rho_1 u_x dy.1.c_1.T_1 - k_1 dy.1.\frac{\partial T_1}{\partial x}$$

$$q_y = \rho_1 u_y dx.1.c_1.T_1 - k_1 dx.1.\frac{\partial T_1}{\partial y}$$

$$\Longrightarrow$$

$$\rho_1 c_1 u_x dy \frac{\partial T_1}{\partial x}dx - \rho_1 c_1 u_y dx \frac{\partial T_1}{\partial y}dy + k_1 dy \frac{\partial^2 T_1}{\partial x^2}dx + k_1 dx.1.\frac{\partial^2 T_1}{\partial y^2}dy =$$

$$\rho_1 c_1 dx\, dy \frac{\partial T_1}{\partial t}$$

$$\Rightarrow \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = \frac{1}{\alpha_1}\left[u_x \frac{\partial T_1}{\partial x} + u_y \frac{\partial T_1}{\partial y} + \frac{\partial T_1}{\partial t}\right]$$

$$u_y = 0, u_x = U_\infty,$$

فرض می‌کنیم که نفوذ حرارتی در جهت x در مقایسه با حرکت توده قابل صرف‌نظر است

$$\theta_1 = T_1 - T_\infty \Rightarrow \frac{\partial^2 \theta_1}{\partial y^2} = \frac{1}{\alpha_1}\left[U_\infty \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_1}{\partial t}\right]$$

برای صفحه:

$$\theta_2 = T_2 - T_\infty \Rightarrow \frac{\partial^2 \theta_2}{\partial y^2} = \frac{1}{\alpha_2}\frac{\partial \theta_2}{\partial t}$$

$$IC \begin{cases} at\ t = t_p: \theta_1(x,y,t_p) = 0 \\ at\ t = t_p, y = 0: \theta_2(0,t_p) = 0 \end{cases}$$

$$BC \begin{cases} at\ y = 0: \theta_1(x,0,t) = \theta_2(0,t),\ \ k_1 \frac{\partial \theta_1}{\partial y} = k_2 \frac{\partial \theta_2}{\partial y} \\ at\ y = \delta(x,t): \frac{\partial \theta_1}{\partial y} = 0, \theta_1(x,\delta,t) = 0 \\ at\ y = -\delta^*: q'' = -k_2 \frac{\partial \theta_2}{\partial y} \\ at\ x = 0: \theta_1(0,y,t) = 0 \end{cases}$$

$$\theta_1(x,y,t) = a_0 + a_1 y + a_2 y^2,\ \theta_2(y,t) = b_0 + b_1 y + b_2 y^2$$

$$\left.\frac{\partial\theta_2}{\partial y}\right|_{y=-\delta^*}=-\frac{q''}{k_2}=b_1-2b_2\delta^*\ ,\ \theta_1(x,0,t)=\theta_2(0,t)\Rightarrow a_0=b_0$$

$$k_1\left.\frac{\partial\theta_1}{\partial y}\right|_{y=0}=k_2\left.\frac{\partial\theta_2}{\partial y}\right|_{y=0}\Rightarrow k_1a_1=k_2b_1\ ,\left.\frac{\partial\theta_1}{\partial y}\right|_{y=\delta}=0\Rightarrow a_1=-2a_2\delta$$

$$\theta_1(x,\delta,t)=0\Rightarrow a_0=a_2\delta^2\Rightarrow\theta_1(x,y,t)=a_2\delta^2\left[1-\frac{2y}{\delta}+\left(\frac{y}{\delta}\right)^2\right]$$

$$a_0=b_0=a_2\delta^2\ ,b_1=\frac{k_1}{k_2}a_1=\frac{k_1}{k_2}(-2a_2\delta)\ ,\ b_2=\frac{q''}{2k_2\delta^*}-\frac{k_1}{k_2\delta^*}a_2\delta^2$$

$$\theta_2(y,t)=a_2\delta^2\left[1-2\frac{k_1}{k_2}\frac{y}{\delta}-\frac{k_1}{k_2\delta^*}\left(\frac{y}{\delta}\right)^2\right]+\frac{q''}{2k_2\delta^*}y^2$$

$$\Rightarrow\int_0^\delta\theta_1dy=a_2\delta^2\left[y-\frac{y^2}{\delta}-\frac{y^3}{3\delta^2}\right]_0^\delta=a_2\frac{\delta^3}{3}$$

$$\int_{-\delta^*}^0\theta_2dy=-a_2\delta^2\left[-\delta^*-\frac{k_1}{k_2}\frac{\delta^{*2}}{\delta}+\frac{k_1}{k_2}\frac{\delta^{*3}}{3\delta}\right]+\frac{q''}{2k_2}\frac{\delta^{*2}}{3}$$

$$\Rightarrow\frac{1}{\alpha_1}\left[U_\infty\frac{\partial}{\partial x}\left(a_2\frac{\delta^3}{3}\right)+\frac{\partial}{\partial t}\left(a_2\frac{\delta^3}{3}\right)\right]=2a_2\delta$$

$$\left.\frac{\partial\theta_2}{\partial y}\right|_{y=0}-\left.\frac{\partial\theta_2}{\partial y}\right|_{y=-\delta^*}=\frac{1}{\alpha_2}\frac{\partial}{\partial t}\int_{-\delta^*}^0\theta_2dy$$

$$\Rightarrow\frac{-q''}{k_2}+2a_2\delta\frac{k_1}{k_2}=\frac{1}{\alpha_2}\frac{\partial}{\partial t}\left[-a_2\delta^2\left(-\delta^*-\frac{2}{3}\frac{k_1}{k_2}\right)+\frac{q''}{2k_2}\frac{\delta^{*2}}{3}\right]$$

می‌دانیم که $a_2=f(\delta)$، بنابراین با استفاده از این دو معادله $\delta(x,t)$ و $a_2=f(\delta)$ به دست خواهد آمد:

$$\text{فرض}:f(\delta)=a_2=\frac{A}{\delta},A=cte:\frac{1}{\alpha_1}\left[U_\infty\frac{\partial}{\partial x}\left(\frac{A\delta^2}{3}\right)+\frac{\partial}{\partial t}\left(\frac{A\delta^2}{3}\right)\right]=2A\Rightarrow$$

$$U_\infty\int_0^x\frac{\partial}{\partial x}\delta^2dx+\frac{\partial}{\partial t}\int_0^x\delta^2dx=6\,\alpha_1\int_0^x dx$$

فرض: $\delta(x,t)=A_0x+A_1$

$$at\ x=0:\delta=0\Rightarrow\delta(x,t)=A_0x\Rightarrow U_\infty(A_0x)^2+\frac{\partial}{\partial t}\left(A_0^2\frac{x^2}{3}\right)=6\alpha_1x$$

$$\Rightarrow A_0^2=u=\frac{6\alpha_1}{U_\infty x}\left[1-\exp\left(\frac{-3U_\infty}{x}t\right)\right]$$

$$\Rightarrow\delta(x,t)=\sqrt{\frac{6\alpha_1}{U_\infty}x\left[1-\exp\left(\frac{-3U_\infty}{x}t\right)\right]}\ ,\ a_2=\frac{A}{\delta(x,t)a}$$

فصل سوم

مسائل یک بعدی پایا، توابع بسل

مساله ۱-۳

(a

$$q=\frac{T_1-T_2}{\frac{1}{k}\int_{s_1}^{s_2}\frac{ds}{A(s)}}\quad,\quad q=k\bar{A}\,\frac{T_1-T_2}{s_2-s_1}\qquad(1)$$

کارتزین $\bar{A}=A$ استوانه‌ای $\bar{A}=\frac{A_2-A_1}{Ln\left(\frac{A_2}{A_1}\right)}$ کروی $\bar{A}=(A_1A_2)^{\frac{1}{2}}$

$$q=\frac{T_1-T_2}{\frac{1}{k}\int_{x_1}^{x_2}\frac{dx}{A=cte}}=kA\frac{T_1-T_2}{x_2-x_1}\rightarrow\bar{A}=A\quad \text{مختصات کارتزین}$$

$$q=\frac{T_1-T_2}{\frac{1}{k}\int_{r_1}^{r_2}\frac{dr}{2\pi rL}}=\frac{2\pi k(T_1-T_2)}{Ln\left(\frac{r_2}{r_1}\right)}\qquad(2)\ \text{مختصات استوانه‌ای}$$

$$(1)\,,(2)\ \Rightarrow\frac{2\pi kL(T_1-T_2)}{Ln\left(\frac{r_2}{r_1}\right)}=k\bar{A}\,\frac{T_1-T_2}{r_2-r_1}\Rightarrow\bar{A}=\frac{2\pi L(r_2-r_1)}{Ln\left(\frac{r_2}{r_1}\right)}=\frac{A_2-A_1}{Ln\left(\frac{A_2}{A_1}\right)}$$

$$q=\frac{T_1-T_2}{\frac{1}{k}\int_{r_1}^{r_2}\frac{dr}{4\pi r^2}}=\frac{4\pi k(T_1-T_2)}{\frac{1}{r_1}-\frac{1}{r_2}}\ .\ (3)\ \text{مختصات کروی}$$

$$(1)\,,(3)\ \Rightarrow\frac{4k\pi(T_1-T_2)}{\frac{1}{r_1}-\frac{1}{r_2}}=\frac{k\bar{A}(T_1-T_2)}{r_2-r_1}\Rightarrow\bar{A}=\frac{4\pi(r_2-r_1)}{\frac{r_2-r_1}{r_1r_2}}=4\pi r_1r_2$$

$$A_1 = 4\pi r_1^2 \ , A_2 = 4\pi r_2^2 \ \Rightarrow \ (A_1A_2)^{\frac{1}{2}} = 4\pi r_1 r_2$$

b)

مختصات استوانه‌ای: $\bar{A} = \frac{A_2 - A_1}{ln\left(\frac{A_2}{A_1}\right)} \quad \lim_{A_2 - A_1} \frac{A_2 - A_1}{ln\left(\frac{A_2}{A_1}\right)} \xrightarrow{H} \frac{1-0}{\frac{1}{A_2}} = A_2$

$$\lim_{A_2 \to A_1} A_2 = \lim_{r_2 \to r_1} 2\pi r_2 L = \pi L \lim_{\varepsilon \to 0}(r_1 + \varepsilon + r_1) = \pi L(r_2 + r_1)$$

$$\bar{A} = \pi L(r_2 + r_1) = \frac{2\pi l(r_2 + r_1)}{2} = \frac{A_1 + A_2}{2}$$

مختصات کروی: $\bar{A} = \sqrt{A_1 A_2}\, , (a-b)^2 = a^2 + b^2 - 2ab$

$$\Rightarrow 2ab = a^2 + b^2 - (a-b)^2 \ (1)$$

$$a = \sqrt{A_1} \ , b = \sqrt{A_2} \ (2)$$

$$(1), (2) \Rightarrow 2\sqrt{A_1}\sqrt{A_2} = A_1 + A_2 - \left(\sqrt{A_1} - \sqrt{A_2}\right)^2$$

$$\Rightarrow \bar{A} = \frac{A_1 + A_2}{2} - \frac{\left(\sqrt{A_1} - \sqrt{A_2}\right)^2}{2} \ \Rightarrow \ \lim_{A_1 \to A_2} \bar{A} = \frac{A_1 + A_2}{2}$$

مساله ۲-۳)

a)

تولید انرژی به وسیله الکتریسیته $\quad q_e = \frac{v^2}{R} = R_i^2$

$i = \frac{\Delta E}{R_e} \ \Rightarrow \ i = k_e \frac{dE}{dy} \qquad R_e:$ مقاومت الکتریکی $= \frac{1}{k_e}$

$$\Rightarrow \ q_e = R_e \left(\frac{dE}{dy}\right)^2$$

موازنه انرژی برای سیستم:

$$wl(q_x - q_{x+dx}) + q_e wld_x = 0 \ \Rightarrow \ -\frac{dq_x}{d_x} d_x + q_e wld_x = 0$$

$$\Rightarrow \ -\frac{dq_x}{d_x} + q_e = 0 \ \Rightarrow \ \frac{d}{d_x}\left(k \frac{dT}{dx}\right) + R_e \left(ke \frac{dE}{dy}\right)^2 = 0$$

$$\Rightarrow \ \frac{d}{d_x}\left(k \frac{dT}{d_x}\right) + k_e \left(\frac{dE}{dy}\right)^2 = 0$$

b)

$$(q_x - q_{x+dx})d_x d_y + (q_y - q_{y+dy})d_z d_x - (q_z - q_{z+dz})d_y d_z$$
$$+(q_{ex} + q_{ey} + q_{ez})d_x d_y d_z = 0$$

$$\Rightarrow -\frac{dq_x}{d_x} - \frac{dq_y}{d_y} - \frac{dq_z}{d_y} + R_e\left(ke^2\left(\frac{dE}{dx}\right)^2 + ke^2\left(\frac{dE}{dy}\right)^2 + ke^2\left(\frac{dE}{dz}\right)^2\right) = 0$$

$$\underbrace{\frac{d}{d_x}\left(k\frac{dT}{d_x}\right) + \frac{d}{dy}\left(k\frac{dT}{d_y}\right) + \frac{d}{d_z}\left(k\frac{dT}{d_z}\right)}_{\nabla(k\nabla T)} = ke\underbrace{\left[\left(\frac{dE}{d_x}\right)^2 + \left(\frac{dE}{d_y}\right)^2 + \left(\frac{dE}{d_z}\right)^2\right]}_{\nabla E^2}$$

$$= \nabla k(\nabla T) + k_e(\nabla E^2) = 0$$

مساله ۳-۳)

a)

برای سیستم ۱: $\quad -\frac{d}{d_x}\left(k_1 A \frac{dT_1}{d_x}\right) = 0 \ \ A = cte \ , \ k_1 = cte$

$$\Rightarrow \ \frac{d^2T_1}{dx^2} = 0 \ \Rightarrow \ T_1 = Ax + B$$

برای سیستم ۲: $\quad -\frac{d}{d_x}\left(k_2 A \frac{dT_2}{d_x}\right) = 0 \ \Rightarrow \ \frac{d^2T_2}{dx^2} = 0 \ \Rightarrow \ T_2 = Cx + D$

شرایط مرزی: $\quad$ (I) $\quad k_1 \frac{dT_1(0)}{d_x} = h(T_1(0) - T_\infty) + q''$

(II) $\quad T_1(L_1) = T_2(0) \ , \ k_1 \frac{dT_1(L_1)}{d_x} = k_2 \frac{dT_2(0)}{d_x}$

(III) $\quad k_2 \frac{dT_2(L_2)}{d_x} = h(T_2(L_2) - T_\infty)$

$$T_1 = Ax + B \ \Rightarrow \ I: \ \Rightarrow \ k_1 A = h(B - T_\infty) + q' \ \Rightarrow \ B = \frac{k_1 A}{h} + T_\infty - \frac{q''}{h}$$

$$T_2 = Cx + D \ \Rightarrow \ II: \ \Rightarrow \ AL_1 + B = D \ \Rightarrow \ D = AL_1 + \frac{k_1 A}{h} + T_\infty - \frac{q''}{h}$$

$$(II) \Rightarrow k_1 A = k_2 C \ \Rightarrow \ C = \frac{k_1 A}{k_2} \ \ , (III) \ \Rightarrow \ \ k_2 C = h(CL_2 + D - T_\infty)$$

بعد از حل چهار معادله و چهار مجهول خواهیم داشت:

$$A = \frac{q'' k_2}{k_1 L_2 h + hk_2} \, , B = \frac{q'' k_2 k_1}{h(k_1 L_2 h + hk_2)} + T_\infty - \frac{q''}{h}$$

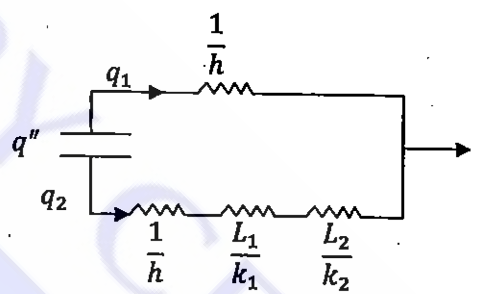
$$\Rightarrow T_1 = \frac{q'' k_2 x}{k_1 L_2 h + h k_2} + \frac{q'' k_2 k_1}{h(k_1 L_2 h + h k_2)} + T_\infty - \frac{q''}{h}$$

$$C = \frac{q'' k_1}{k_1 L_2 h + h k_2}, D = \left(L_1 + \frac{k_1}{h}\right)\left(\frac{q'' k_2}{k_1 L_2 h + h k_2}\right) + T_\infty - \frac{q''}{h}$$

$$\Rightarrow T_2 = \frac{q'' k_1 x}{k_1 L_2 h + h k_2} + \left(L_1 + \frac{k_1}{h}\right)\left(\frac{q'' k_2}{k_1 L_2 h + h k_2}\right) + T_\infty - \frac{q''}{h}$$

$$\Rightarrow q = k_2 A \frac{dT_2(L_2)}{d_x} = k_2 A \frac{q'' k_1}{k_1 L_2 h + h k_2}$$

(b



مساله ۵-۳)

در جهت r فرمولاسیون متمرکز در نظر می‌گیریم و در جهت x فرمولاسیون گسترده.

برای سیال داخل لوله:

$$\dot{m}h_i - \dot{m}h_o - q_{conv} = 0$$

$$\pi R^2 \left(\rho_v c_p T_x - \rho v c_p T_{x+dx}\right) - h_i (T - T_W) 2\pi R d_x = 0$$

$$\frac{dT}{dx} + \frac{2h_i}{R\rho v c_p}(T - T_W) = 0 \Rightarrow \frac{dT}{dx} + m(T - T_W) = 0 \ (1)$$

برای دیواره لوله: به دلیل اینکه δ کوچک است، دمای دیواره داخلی و خارجی را T_W در نظر می‌گیریم.

$$h_i A_i (T - T_W) + u''' dv = h_0 A_0 (T_W - T_\infty)$$

$$h_i 2\pi R d_x (T - T_W) + u''' 2\pi R \delta d_x = h_0 2\pi (R + \delta)(T_W - T_\infty) d_x$$

$$h_i (T - T_W) + u''' \delta = h_0 (T_W - T_\infty)\left(1 + \frac{\delta}{R}\right)$$

$$h_i T - h_i T_W - h_0 T_W - h_0 T_W \frac{\delta}{R} + h_0 T_\infty + h_0 T_\infty \frac{\delta}{R} + u''' \delta = 0$$

$$T_W \left(h_i + h_0 + \frac{h_0 \delta}{R}\right) = h_i T + h_0 T_\infty + h_0 T_\infty \frac{\delta}{R} + u''' \delta$$

$$\Rightarrow T_W = \frac{h_i T + h_0 T_\infty \left(1 + \frac{\delta}{R}\right) + u''' \delta}{h_i + h_0 \left(1 + \frac{\delta}{R}\right)} \Rightarrow T_W = C + DT$$

$$(1) \Rightarrow \frac{dT}{dx} + m(T - C - DT) = 0 \Rightarrow \frac{dT}{dx} + m(1 - D)T = mC$$

$$\Rightarrow T(x) = \frac{C}{1-D} + \propto \exp(-m(1-D)x)$$

$$at\ x = 0 \Rightarrow T = T_i \Rightarrow \propto = T_i - \frac{C}{1-D}$$

$$\Rightarrow T(x) = \frac{C}{1-D} + \left(T_i - \frac{C}{1-D}\right)\exp(-m(1-D)x)$$

a) اگر $h_i = h_0 = h \Rightarrow D = \frac{h}{h\left(2 + \frac{\delta}{R}\right)} = \frac{R}{2R + \delta}$, $C = \frac{h T_\infty \left(1 + \frac{\delta}{R}\right) + u''' \delta}{h\left(2 + \frac{\delta}{R}\right)}$

b) اگر $h_i \gg h_0 \Rightarrow D = 1$, $C = \frac{u''' \delta}{h_i} \Rightarrow T_W = \frac{u''' \delta}{h_i} + T$

$$\Rightarrow \frac{dT}{dx} = \frac{m u'''}{h_i} \Rightarrow T(x) = \frac{m u'''}{h_i} x + c_1$$

$$at\ x = 0 \Rightarrow T = T_i \Rightarrow T(x) = \frac{m u'''}{h_i} x + T_i \quad , \quad m = \frac{2h_i}{R\rho v c_p}$$

$$\Rightarrow T(x) = \frac{2u'''}{R\rho v c_p} x + T_i$$

مساله ۶-۳)

$$u''' = u_0''' \sin\pi\left(\frac{a}{A}\right)$$

$$Q_{total} = u_0''' \int_0^A \sin\left(\frac{\pi a}{A}\right) da = -u_0''' \frac{A}{\pi} \cos\left(\frac{\pi a}{A}\right)\Big|_0^A = \frac{2u_0''' A}{\pi}$$

$$Q_{local} = u_0''' \int_0^a \sin\left(\pi \frac{a}{A}\right) da = -u_0''' \frac{A}{\pi} \cos\left(\frac{\pi a}{A}\right)\Big|_0^a = \left[1 - \cos\left(\frac{\pi a}{A}\right)\right] \frac{u'''_0 A}{\pi}$$

$$\frac{Q_{local}}{Q_{total}} = \frac{\omega c\left(T_c - T_{c_i}\right)}{\omega c\left(T_{c_0} - T_{c_i}\right)} = \frac{\left(\frac{u_0''' A}{\pi}\right)}{2\frac{u_0''' A}{\pi}}\left[1 - \cos\left(\frac{\pi a}{A}\right)\right] \Rightarrow \frac{\left(T_c - T_{c_i}\right)}{\left(T_{c_0} - T_{c_i}\right)} = \frac{1}{2}\left[1 - \cos\left(\frac{\pi a}{A}\right)\right]$$

$$Q_{total} = \omega c\left(T_{c_o} - T_{c_i}\right) = \frac{2u_0''' A}{\pi} \Rightarrow T_{c_o} - T_{c_i} = \frac{2u_0''' A}{\pi\omega c}$$

شار : $q'' = u_0''' \sin\left(\frac{\pi a}{A}\right) = h(T_\omega - T_c) = h\left(T_\omega - T_{c_i}\right) - h\left(T_c - T_{c_i}\right)$

$$\Rightarrow T_\omega - T_{c_i} = 2u_0''' \sin\left(\frac{\pi a}{A}\right) / h + T_c - T_{c_i}$$

$$\Rightarrow \frac{\left(T_\omega - T_{c_i}\right)}{\left(T_{c_o} - T_{c_i}\right)} = \frac{T_c - T_{c_i}}{T_{c_o} - T_{c_i}} + \frac{2u_0''' \sin\left(\frac{\pi a}{A}\right)}{T_{c_o} - T_{c_i}} \xrightarrow{T_{c_o} - T_{c_i} = \frac{2u_0''' A}{\pi\omega c}}$$

$$\Rightarrow \frac{\left(T_\omega - T_{c_i}\right)}{\left(T_{c_o} - T_{c_i}\right)} = \frac{1}{2}\left[1 - \cos\left(\frac{\pi a}{A}\right)\right] + \frac{\pi}{2}\left(\frac{\omega c}{Ah}\right)\sin\left(\frac{\pi a}{A}\right)$$

موازنه حرارتی برای میله:

تجمع = مصرف - تولید + خروجی - ورودی $\Rightarrow \frac{1}{2}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{u_0''' \sin\left(\frac{\pi a}{A}\right)}{k} = 0$

$$\Rightarrow T = -\frac{u_0''' \sin\left(\pi\frac{a}{A}\right)}{4k} r^2 + c_1 Lnr + c_2$$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \Rightarrow C_1 = 0 \quad , \text{ at } r = 0 : T - T_0 \Rightarrow C_2 = T_0$$

$$\Rightarrow T = -\frac{u_0''' \sin\left(\pi\frac{a}{A}\right)}{4k} r^2 + T_0$$

$$\text{at } r = R : T = T_\omega \Rightarrow T_\omega = -\frac{u_0''' \sin\left(\frac{\pi a}{A}\right)}{4k} R^2 + T_0$$

$$T_0 - T_\omega = \frac{u_0''' \sin\left(\frac{\pi a}{A}\right)}{Ak} R^2$$

$$\left(T_o - T_{c_i}\right) - \left(T_\omega - T_{c_i}\right) = \frac{u_0''' \sin\left(\frac{\pi a}{A}\right) R^2}{4k}$$

$$\Rightarrow \frac{T_o - T_{c_i}}{T_{c_o} - T_{c_i}} - \frac{T_\omega - T_{c_i}}{T_{c_o} - T_{c_i}} = \frac{u_0''' \sin\left(\frac{\pi a}{A}\right)}{4k\left(T_{c_o} - T_{c_i}\right)} R^2$$

$$\frac{T_o - T_{c_i}}{T_{c_o} - T_{c_i}} = \frac{1}{2}\left[1 - \cos\left(\frac{\pi a}{A}\right)\right] + \frac{\pi}{2}\left(\frac{\omega c}{hA}\right)\sin\left(\frac{\pi a}{A}\right) + \frac{\pi}{2}\omega c\frac{R^2}{4k}\sin\left(\frac{\pi a}{A}\right)$$

مسئله ۷-۳)

$$\dot{E}_{in} - \dot{E}_{ant} + \dot{E}_{gen} - \dot{E}_{con} = \frac{dE}{dt}$$

$$\left(q_r A_r - q_{r+dr} A_{r+dr}\right) + \left[(\mu p \omega r)(2\pi r dr)\right] - 0 = \frac{d(dmcT)}{dt}$$

$$\Rightarrow -\frac{d}{dr}\left(q_r A_r\right)dr + \left[(\mu p \omega r)(2\pi r dr)\right] = \underbrace{dm}_{\rho_1 dv}\, c_1 \frac{dT}{dt}$$

$$\Rightarrow -\frac{d}{dr}\left(-k_1\frac{dT}{dr}\, 2\pi r\, \delta_1\right)\frac{dr}{r} + \left[(\mu p \omega r)(2\pi r dr)\right] = \rho_1 c_1 \left(2\pi r dr \delta_1\right)\frac{dT}{dt}$$

$$\Rightarrow \frac{dT}{dt} = \frac{k_1\delta_1}{\rho_1 c_1 \delta_1} \times \frac{1}{r} \times \frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\mu p \omega r}{\rho_1 c_1 \delta_1}$$

در این مساله مقادیر متوسط را برای پارامترها در نظر می‌گیریم:

$$\bar{a} = \frac{\overline{k\delta}}{\overline{\rho c \delta}} \quad , \quad \overline{k\delta} = \frac{k_1\delta_1 + k_2\delta_2}{2} \quad , \quad \overline{\rho c \delta} = \frac{\rho_1 c_1 \delta_1 + \rho_2 c_2 \delta_2}{2}$$

$$\Rightarrow \frac{dT}{dt} = \frac{\bar{a}}{r}\ \frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\mu p w r}{2\overline{\rho c \delta}} \qquad IC:\ T(r,0) = T_\infty$$

$$\bar{k} = \frac{k_1 + k_2}{2} \quad , \quad \bar{h} = \frac{h_3 + h_4}{2} \qquad BC\begin{cases} \frac{dT}{dr}(0,t) = 0 \\ -\bar{k}\frac{dT(R,t)}{dr} = \bar{h}\left(T(R,t) - T_\infty\right) \end{cases}$$

مساله ۸-۳)

برای سیال درون لوله:

$$A_r\left(q_r - q_r + d_r\right) + A_z\left(q_z - q_z + d_z\right) = 0$$

$$\Rightarrow 2\pi r dz q_r - 2\pi r dz q_r + d_r\left(\rho v c_p T_z - \rho v c_p T_z + d_z\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(kr\frac{dT_b}{dr}\right) + \rho v c_p r \frac{dT_b}{dr} = 0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dT_b}{dr}\right) + \frac{v}{\alpha}\frac{dT_b}{dz} = 0 \text{ (i)}$$

T_b معیاری از دمای سیال بوده و از اختلاف بین دمای سیال و دیواره لوله حاصل می‌شود:

$$BC\begin{cases} z = 0: T_b(r,0) = T_i - T_w \\ r = 0: \frac{dT_b}{dr} = 0 \\ r = R_i: T_b(R_i, z) = 0 \end{cases} \qquad , T_b = T - T_w$$

برای دیواره لوله:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_w}{dr}\right)+\frac{u'''}{k_w}=0,\ \text{BC}\begin{cases} r=R_0 \Longrightarrow \frac{dT_w}{dr}=0\ (ii) \\ r=R_i \Longrightarrow T_w = T_{wi}\quad (iii)\end{cases}$$

$$\Longrightarrow \frac{d}{dr}\left(r\frac{dT_w}{dr}\right)=\frac{-u'''r}{k_w} \Longrightarrow \frac{dT_w}{dr}=\frac{-u'''r}{2k_w}+\frac{C_1}{r} \Longrightarrow T_w(r)=\frac{-u'''r^2}{4k_w}+C_1 Lnr+ C_2$$

$$(ii): -\frac{2u'''R_0}{4k_w}+\frac{C_1}{R_0}=0 \Longrightarrow C_1=\frac{u'''R_0^2}{2k_w}$$

$$(iii): -\frac{u'''R_i^2}{4k_w}+\frac{u'''R_0^2}{2k_w}\text{Ln}(R_i)+C_2 = T_{wi}$$

$$\Longrightarrow C_2=\frac{u'''R_i^2}{4k_w}-\frac{u'''R_i^2}{2k_w}\text{Ln}(R_i)+T_{wi}$$

$$\Longrightarrow T_w(r)=\frac{-u'''r^2}{4k_w}+\left(\frac{u'''R_0^2}{2k_w}\right)Lnr+\frac{u'''R_i^2}{4k_w}-\frac{u'''R_i^2}{2k_w}\text{Ln}(R_i)+T_{wi}$$

$$\Longrightarrow T_w(R_0)=T_{wo}=\frac{-u'''r^2}{4k_w}+\left(\frac{u'''R_0^2}{2k_w}\right)Lnr+\frac{u'''R_i^2}{4k_w}-\frac{u'''R_i^2}{2k_w}\text{Ln}(R_i)+T_{wi}$$

برای معادله دیفرانسیل جزیی (i):

$$T_b(r,z)=R(r).Z(z)$$

$$\frac{dT_b}{dr}=ZR' \qquad \frac{d^2T_b}{dr^2}=ZR'' \qquad \frac{dT_b}{dz}=RZ'$$

$$(i) \Longrightarrow ZR''+\frac{1}{r}ZR'+\frac{\overset{\propto'}{\widehat{V}}}{\propto}RZ'=0$$

$$\Longrightarrow \frac{R''}{R}+\frac{1}{r}=\frac{R'}{R}=-\frac{R''}{R}\frac{RZ'}{Z}=-\lambda^2 \Longrightarrow$$

$$\begin{cases} R''+\frac{1}{r}R'+\lambda^2R=0 \Longrightarrow r^2R''+rR'+\lambda^2\lambda^2R=0 \\ Z'-\frac{\lambda^2}{\propto'}Z=0 \Longrightarrow Z(z)=c\,exp(-\propto'\lambda^2Z)\end{cases}$$

$$R(r)=AJ_0(\lambda r)+BY_0(\lambda r)$$

at $\quad r=0 \quad T_b=finite \Longrightarrow R(r=0)=finite \Longrightarrow B=0$

at $\quad r=R_i \quad T_b=0 \Longrightarrow R=0 \Longrightarrow AJ_0(\lambda R_i)=0$

با استفاده از این معادله λ_n به دست خواهد آمد $\Longrightarrow J_0(\lambda_n R_i)=0,\ \ n=1,2,\dots,\infty$

$$T_b(r,z)=\sum_{n=1}^{\infty}A_ne^{-\propto'\lambda_n^2z}J_0(\lambda_nr)\xrightarrow{z=0 \text{ در ابتدای لوله}} T_i-T_w=\sum_{n=1}^{\infty}J_n(\lambda_nr)A_n$$

$$\overset{rJ_0(\lambda_m r)}{\Longrightarrow}\int_0^{R_i}(T_i-T_w)rJ_0(\lambda_mr)dr=\int_0^{R_i}A_nr(\lambda_nr)J_0(\lambda_mr)dr,\ m=n$$

$$\Longrightarrow \int_0^{R_i}rJ_i^2(\lambda_nr)dr=\frac{R_i^2}{2}J_1^2(\lambda_nR_i),\ \int_i^{R_i}rJ_0(\lambda_nr)d_r=J_1(\lambda_nR_i)R_i$$

$$A_n=\frac{(T_i-T_w)J_1(\lambda_nR_i)}{\frac{R_i^2}{2}J_1^2(\lambda_nR_i)}=\frac{(T_i-T_w)}{R_iJ_1(\lambda_nR_1)}$$

$$\Longrightarrow\ T_b(r,z)=\sum_{n=1}^{\infty}\frac{2(T_i-T_w)}{R_iJ_1(\lambda_nR_i)}\ J_0(\lambda_nr)e^{-\propto'\lambda_n^2z}$$

مساله ۹-۳)

(a

$$q_r|_r-q_r|_{r+d_r}=\rho c_p(4\pi r^2)dr\frac{\partial T}{\partial t}$$

$$\Longrightarrow -\frac{d}{dr}\left(-k\frac{dT}{dr}4\pi r^2\right)dr=\rho c_p4\pi r^2dr\frac{dT}{dt}$$

$$\Longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=\frac{1}{\propto}\frac{dT}{dt}$$

$$BC\begin{cases} at\quad R=R_i \ :\ q''=k\frac{dT}{dt} \\ at\quad R=R_0\ :\ h(T-T_\infty)=k\frac{dT}{dr}\end{cases} \qquad IC: t=0:\quad T=T_\infty$$

مبدا مختصات را در سطح داخلی پوسته در نظر می‌گیریم و ضخامت پوسته را δ فرض می‌کنیم:

شکل انتگرالی: $\int_0^{\delta}\frac{\partial}{\partial t}\left(r^2\frac{\partial\theta}{\partial r}\right)dr=\frac{1}{\propto}\int_0^{\delta}r^2\frac{\partial\theta}{\partial t}dr \qquad \theta=T-T_\infty$

$$r^2\frac{d\theta}{dt}\Big|_{\delta}-r^2\frac{d\theta}{dt}\Big|_0=\frac{1}{\propto}\int_0^{\delta}r^2\frac{\partial\theta}{\partial t}d_r$$

فرض: $\theta=(r^2+b_1r+b_2)\,\tau(t)$

$$\begin{cases} R=0 \quad \frac{d\theta}{dt}=\frac{q''}{k} \\ R=\delta\ \ h\theta=k\frac{d\theta}{dr}\end{cases} \longrightarrow b_1\tau(t)=\frac{q''}{k}\Longrightarrow b_1=\frac{q''}{k\tau(t)}$$

$$\Longrightarrow\left(\delta^2+\frac{q''\delta}{k\tau(t)}+b_2\right)=\frac{k}{h}\left(2\delta+\frac{q''}{k\tau(t)}\right)$$

$$\Longrightarrow b_2=\frac{2k\delta}{h}+\frac{q''}{h\tau(t)}-\delta^2-\frac{q''\delta}{k\tau(t)}\quad\Longrightarrow$$

$$\theta=\left[r^2+\frac{q''}{h\tau(t)}r+\left(\frac{2k\delta}{h}+\frac{q''}{h\tau(t)}-\delta^2-\frac{q''\delta}{k\tau(t)}\right)\right]\tau(t)$$

$$\Longrightarrow \tau(t)\delta^2\left[2\delta+\frac{q''}{k\tau(t)}\right]=\frac{1}{\alpha}\int_0^{\delta} r^2\frac{d\theta}{d\tau(t)}.\frac{d\tau(t)}{dt}d_r$$

$$\Longrightarrow \tau(t)\delta^2\left[2\delta+\frac{q''}{k\tau(t)}\right]=\frac{1}{\alpha}\int_0^{\delta} r^2\left[r^2+\frac{2k\delta}{h}-\delta^2\right]d_r\frac{d\tau(t)}{dt}$$

$$\Longrightarrow 2\tau(t)\delta^3+\frac{q''\delta^2}{k}=\frac{1}{\alpha}\left[\frac{\delta^5}{5}+\frac{2k\delta^4}{3h}-\frac{\delta^5}{3}\right]\frac{d\tau(t)}{dt}$$

$$\Longrightarrow \frac{Ln\left(2\delta^3\tau(t)+\frac{q''}{k}\delta^2\right)}{2\delta^3}=\frac{\alpha t}{\frac{\delta^5}{5}+\frac{2k\delta^4}{3h}-\frac{\delta^5}{3}}+C$$

$$t=0\Longrightarrow\theta(r,0)=0\Longrightarrow\left[r^2+\frac{2kr}{h}-\delta^5\right]\tau(t)+\frac{q''r}{k}+\frac{q''}{h}-\frac{q''\delta}{k}=0$$

$$\Longrightarrow \tau(t)=\frac{q''\left(\frac{\delta}{k}-\frac{r}{k}-\frac{1}{h}\right)}{r^2+\frac{2k\delta}{h}-\delta^2}\quad \text{سپس}\quad C=\frac{\ln\left[\frac{2\delta^3q''\left(\frac{\delta}{k}-\frac{r}{k}-\frac{1}{h}\right)}{r^2+\frac{2k\delta}{h}-\delta^2}\right]+\frac{q''\delta^2}{k}}{2\delta^3}$$

$$\Longrightarrow Ln\frac{\left[2\delta^3\tau(t)+\frac{q''}{k}\delta^2\right]}{\left[\frac{2\delta^3q''\left(\frac{\delta}{k}-\frac{r}{k}-\frac{1}{h}\right)}{r^2+\frac{2k\delta}{h}-\delta^2}+\frac{q''\delta^2}{k}\right]}=\frac{2\alpha t}{\frac{\delta^2}{5}+\frac{2}{3}\frac{k}{h}\delta^2-\frac{\delta^3}{3}}$$

$$\Longrightarrow \tau(t)=\left[\frac{2\delta^3q''\left(\frac{\delta}{k}-\frac{r}{k}-\frac{1}{h}\right)}{r^2+\frac{2k\delta}{h}-\delta^2}+\frac{q''\delta}{2k}\right]\exp\left[\frac{2\alpha t}{\frac{\delta^2}{5}+\frac{2}{3}\frac{k\delta^2}{h}-\frac{\delta^3}{3}}\right]-\frac{q''}{2k\delta}$$

$$\Longrightarrow\theta=\left[r^2+\frac{2k\delta}{h}-\delta^2\right]\tau(t)+\frac{q''r}{k}+\frac{q''}{h}-\frac{q''\delta}{k}$$

$$\theta=\left[\frac{2k\delta}{h}-\delta^2\right]\left[\frac{q''\left(\frac{s}{k}-\frac{1}{h}\right)}{\frac{2k\delta}{h}-\delta^2}+\frac{q''}{2k\delta}\right]\exp\left[\frac{2\alpha t}{\frac{\delta^2}{5}+\frac{2}{3}\frac{k\delta^2}{h}+\frac{\delta^3}{3}}\right]-\frac{q''}{2k\delta}+\frac{q''}{h}-\frac{q''\delta}{k}$$

$$\longrightarrow\quad \delta=R_0-R_i$$

b)

عبارت نمایی بزرگتر از دیگر عبارت‌هاست بنابراین با توجه به زمان اگر k افزایش یابد، h و θ کاهش می‌یابند، $k\uparrow, h\downarrow\ \Longrightarrow\ \theta\downarrow$

مساله ۱۱-۳)

$$\frac{d^2T}{dx^2}+\frac{u'''}{k}=0$$

$$BC\begin{cases}x=0: & q''_1+k\frac{dT}{dx}=h(T-T_\infty)\\ x=L: & -k\frac{dT}{dx}=q''_2\end{cases}$$

$$\theta=T-T_\infty\Longrightarrow\frac{d^2\theta}{dx^2}=-\frac{u'''}{k},BC\begin{cases}x=0 & h\theta-k\frac{d\theta}{dx}=q''_1\\ x=L & -k\frac{d\theta}{dx}=q''_2\end{cases}$$

شرایط مرزی ناهمگن است بنابراین : $\theta(x)=\theta_1(x)+\theta_2(x)$

$$\frac{d^2\theta_1}{dx^2}+\frac{d^2\theta_2}{dx^2}=-\frac{u'''}{k},BC\begin{cases}x=0: & h\theta_1+h\theta_2-k\frac{d\theta_1}{dx_1}-k\frac{d\theta_2}{dx_1}=q''_1\\ x=L: & -k\frac{d\theta_1}{dx}-k\frac{d\theta_2}{dx}=q''_2\end{cases}$$

$$\Longrightarrow\begin{cases}\frac{d^2\theta_1}{dx^2}=0\ \longrightarrow\begin{cases}x=0 & h\theta_1-k\frac{d\theta_1}{dx_1}=q''_1\\ x=L & -k\frac{d\theta_1}{dx}=0\end{cases}\\ \frac{d^2\theta_2}{dx^2}=-\frac{u'''}{k}\ \longrightarrow\begin{cases}x=0 & h\theta_2-k\frac{d\theta_2}{dx_1}=0\\ x=L & -k\frac{d\theta_2}{dx}=q''_2\end{cases}\end{cases}$$

مساله ۱۲-۳)

a)

$$q=-kA\frac{dT}{dx}\quad\longrightarrow\ \int_{x_1}^{x_2}-\frac{q}{A}d_x=\int_{T_1}^{T_2}kdT$$

$$\longrightarrow-\frac{q}{A}(x_2-x_1)=k_0\int_{T_1}^{T_2}(1+\beta T)=k_0\left[(T_2-T_1)+\frac{\beta}{2}(T_2^2-T_1^2)\right]$$

$$\longrightarrow k_0\left[(T_2-T_1)+\frac{\beta}{2}(T_2-T_1)(T_2+T_1)\right]=k_0[T_2-T_1]\left(\frac{k_1+k_2}{2}\right)=$$

$$-\frac{q}{A}(x_2-x_1)$$

$$= \left(T_2 - T_1\right)\left[\frac{\overbrace{k_0(1+\beta T_1)}^{k_1} + \overbrace{k_0(1+\beta T_2)}^{k_2}}{2}\right] = \left(T_2 - T_1\right)\left(\frac{k_1 + k_2}{2}\right) =$$

$$-\frac{q}{A}\left(x_2 - x_1\right) \Rightarrow q = k_m A \frac{T_1 - T_2}{x_2 - x_1}$$

b) انتگرال‌گیری مستقیم:

$$\frac{d}{d_x}\left(k\frac{dT}{dx}\right) = 0 \Rightarrow k\frac{dT}{dx} = C_1 \Rightarrow$$

$$\int k_0(1+\beta T)dT = \int C_1 dx \longrightarrow \left(T + \frac{\beta T^2}{2}\right) = C_1 x + C_2$$

$$BC\begin{cases} x = x_1: & T = T_1 \\ x = x_2: & T = T_2 \end{cases}$$

$$\Rightarrow C_1 = \frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{x_1 - x_2}, C_2 = \left(T_1 + \frac{\beta T_1^2}{2}\right) - \frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{x_1 - x_2} x_1$$

$$\Rightarrow \left(T + \frac{\beta T^2}{2}\right) = \frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{x_1 - x_2}(x - x_1) + \left(T_1 + \frac{\beta T_1^2}{2}\right)$$

c) روش کانتروویچ

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0, \frac{d}{dx}\left(k_R\frac{d\theta}{dx}\right) = 0 \rightarrow \frac{d^2\theta}{dx^2} = 0, \quad \frac{d\theta}{dx} = \frac{k}{k_R}\frac{dT}{dx}$$

$$\Rightarrow \theta = C_1 x + C_2,$$

$$\theta = \frac{1}{k_R}\int_{T_R}^{T}\left(k_0(1+\beta T)\right)dT \Rightarrow \theta = \frac{k_0}{k_R}\left[(T - T_R) + \frac{\beta}{2}(T^2 - T_R^2)\right] = C_1 x + C_2$$

$$\text{BC}\begin{cases} x = x_1: & T = T_1 \\ x = x_2: & T = T_2 \end{cases} \longrightarrow C_1 = \frac{k_0}{k_R}\frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{x_1 - x_2}$$

$$C_2 = \frac{k_0}{k_R}\left[(T_1 - T_R) + \frac{\beta}{2}(T_1^2 - T_R^2)\right] - \frac{k_0 x_1}{k_R}\frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{x_1 - x_2}$$

$$\theta = \frac{k_0}{k_R}\left[(T - T_R) + \frac{\beta}{2}(T^2 - T_R^2)\right] = \frac{k_0}{k_R}\frac{\left[(T_1 - T_2) + \frac{\beta}{2}(T_1^2 - T_2^2)\right]}{(x_1 - x_2)}(x - x_2) + \frac{k_0}{k_R}\left[(T_1 - T_R) + \frac{\beta}{2}(T_1^2 - T_R^2)\right]$$

مسئله ۱۳-۳)

با استفاده از موازنه انرژی به شکل دیفرانسیلی خواهیم داشت:

$$C\frac{DT}{Dt} = \nabla(k.\nabla T) + u'''$$

با استفاده از روش کانتروویچ: $\theta = \frac{1}{k_R}\int_{T_R}^{T} k(T)dT, \quad \frac{D\theta}{Dt} = \frac{k}{k_R}\frac{DT}{Dt}$

$$\Rightarrow \frac{D\theta}{Dt} = a\nabla^2\theta + \left(\frac{k}{k_R}\right)u''' \qquad T_R = \text{دمای مرجع}$$

موازنه انرژی به شکل دیفرانسیلی:

$$C\frac{DT}{Dt} = \nabla(k.\nabla T) + u''' = \nabla k.\nabla T + k\nabla^2 T + u_0''' \quad \nabla k = \frac{dk}{dT}(\nabla T)$$

$$C\frac{DT}{Dt} = \frac{dk}{dT}(\nabla T)^2 + k\nabla^2 T + u'''$$

با توجه به فرض دوم $0 = \frac{dk}{dT}\left(\frac{dT}{dx}\right)^2 + k(T)\frac{d^2T}{dx^2} + u'''$ و $BC\begin{cases} T(L) = T_\infty \\ \frac{dT(0)}{dx} = 0 \end{cases}$

$$\frac{dT(0)}{dx} = 0 \Rightarrow \frac{dk}{dT}\left(\frac{dT_0}{dx}\right)^2 + k(T_0)\frac{d^2T}{dx^2} = -u''' \Rightarrow \frac{d^2T}{dx^2} = -\frac{u'''}{k(T_0)}$$

با استفاده از مشتق معادله حاکم:

$$2\frac{dk(0)}{dT}\left(\frac{dT}{dx}\right)\frac{d^2T}{dx^2} + k(T)\frac{d^3T}{dx^3} = 0 \Rightarrow \left.\frac{d^3T}{dx^3}\right|_{x=0} = 0$$

$$2\frac{dk}{dT}\left(\frac{dT}{dx}\right)\frac{d^3T}{dx^3} + 2\frac{dk}{dT}\left(\frac{d^2T}{dx^2}\right)^2 + k(T)\frac{d^4T}{dx^4} = 0 \Rightarrow \left.\frac{d^4T}{dx^4}\right|_{x=0} = \frac{-2\left.\frac{dk}{dT}\right|_{T=T_0} u'''^2}{k(T_0)^3}$$

با استفاده از بسط تیلور برای تابع $T(x)$ خواهیم داشت:

$$T(x) \approx T(0) + \frac{1}{1!}\left.\frac{dT}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2T}{dx^2}\right|_{x=0} x^2 + \frac{1}{3!}\left.\frac{d^3T}{dx^3}\right|_{x=0} + \frac{1}{4!}\left.\frac{d^4T}{dx^4}\right|_{x=0} x^4 + ..$$

$$T(x) = T_0 - \frac{u'''}{2k_0}\frac{x^2}{2!} - \frac{2u'''}{k(T_0)^3}\left(\frac{dk}{dT}\right)_{T=T_\infty}\frac{x^4}{4!} + \cdots$$

مساله ۱۴-۳)

$$\frac{1}{r}\frac{d}{dr}\left(k_w r\frac{dT_w}{dr}\right)+u'''=0,\ \mathrm{BC}\begin{cases} r=R_0 \Longrightarrow \frac{dT_w}{dr}=0 \\ r=R_i \Longrightarrow T_w=T_{wi}\end{cases}$$

$$k_w=k(T)=a+bT$$

$$\frac{1}{r}\left[\frac{dk}{dr}.r\frac{dT_w}{dr}+k\frac{dT_w}{dr}+kr\frac{d^2T_w}{dr^2}\right]+u'''=0$$

$$\Longrightarrow \frac{dk}{dT_w}\left(\frac{dT_w}{dr}\right)^2+\frac{k}{r}\frac{dT_w}{dr}+k\frac{d^2T_w}{dr^2}+u'''=0$$

$$r=R_0\Longrightarrow \frac{dT_w}{dr}=0\Longrightarrow \frac{d^2T_w}{dr^2}=\frac{-u'''}{k(T)}$$

$$2\frac{dk}{dT_w}\left(\frac{dT_w}{dr}\right)\frac{d^2T_w}{dr^2}-\frac{k}{r^2}\frac{dT_w}{dr}+\frac{k}{r}\frac{d^2T_w}{dr^2}+k\frac{d^3T_w}{dr^3}=0\Longrightarrow \frac{d^3T_w}{dr^3}=\frac{u'''}{R_0.k(T)}$$

,....

$$T_w(r)=T_w(R_0)-\frac{1}{2!}\frac{u'''}{k(T)}(r-R_0)^2+\frac{1}{3!}\frac{u'''}{R_0.k(T)}(r-R_0)^3+\cdots$$

مساله ۱۵-۳)

فرض: $k=a+bT$

$$\text{G.E: }\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)+u'''=0\quad BC\begin{cases} r=0 & \frac{dT}{dr}=0 \\ r=R & T=T_{12}\end{cases}$$

$$\frac{1}{r^2}\left[\frac{dk}{dr}r^2\frac{dT}{dr}+2kr\frac{dT}{dr}+kr^2\frac{d^2T}{dr^2}\right]+u_0'''\left[1-\left(\frac{r}{R}\right)^2\right]=0$$

$$\frac{dk}{dr}=\frac{dk}{dT}.\frac{dT}{dr}\longrightarrow \frac{1}{r^2}\left[r^2\frac{dk}{dT}\left(\frac{dT}{dr}\right)^2+2kr\frac{dT}{dr}+kr^2\frac{d^2T}{dr^2}\right]+u_0'''\left[1-\left(\frac{r}{R}\right)^2\right]=0$$

$$\Longrightarrow r^2\frac{dk}{dT}\left(\frac{dT}{dr}\right)^2+2kr\frac{dT}{dr}+kr^2\frac{d^2T}{dr^2}+u_0'''\left[1-\left(\frac{r}{R}\right)^2\right]r^2=0$$

$$k=a+bT\longrightarrow \frac{dk}{dT}=b$$

$$\Longrightarrow r^2b\left(\frac{dT}{dr}\right)^2+2kr\frac{dT}{dr}+kr^2\frac{d^2T}{dr^2}+u_0'''\left[1-\left(\frac{r}{R}\right)^2\right]r^2=0$$

$$\Longrightarrow b\left(\frac{dT}{dr}\right)^2+\frac{2k}{r}\frac{dT}{dr}+k\frac{d^2T}{dr^2}+u_0'''\left[1-\left(\frac{r}{R}\right)^2\right]=0$$

$$r=0\Longrightarrow \frac{dT}{dr}=0\Longrightarrow \frac{d^2T}{dr^2}=-\frac{u_0'''}{a+bT_0}$$

مشتقات معادله حاکم:

$$2b\left(\frac{dT}{dr}\right)\left(\frac{d^2T}{dr^2}\right)+\frac{2k}{r}\frac{d^2T}{dr^2}-\frac{2k}{r^2}\frac{dT}{dr}+k\frac{d^3T}{dr^3}-\frac{2u_0'''}{R^2}r=0$$

$$\Longrightarrow \frac{d^3T}{dr^3}=-\frac{2}{r}\frac{u_0'''}{a+bT_0}$$

$$T(r)=T_0+\left(\frac{dT}{dr}\right)r+\left(\frac{d^2T}{dr^2}\right)\frac{r^2}{2!}+\left(\frac{d^3T}{dr^3}\right)\frac{r^3}{3!}+\cdots$$

$$T(r)=T_0+\frac{u_0'''}{a+bT_0}\frac{r^2}{2!}-\frac{2}{r}\frac{u_0'''}{a+bT_0}\frac{r^3}{3!}+\cdots$$

مساله ۱۶-۳)

$$\frac{dT}{dt}=a\nabla^2T+\frac{u'''}{\rho c}$$

انتقال حرارت پایا به دلیل جابه‌جایی $=hPdx(T-T_\infty)$

$$\Longrightarrow \frac{dT}{dt}=\frac{k}{\rho c}\left(\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}+\frac{d^2T}{dz^2}\right)+\frac{u'''}{\rho c}-hP(T-T_\infty)$$

فرضیات:

۱- در پره تولید انرژی وجود ندارد بنابراین: $\frac{u'''}{\rho c}=0$

۲- انتقال حرارت تنها در جهت x صورت می‌گیرد بنابراین: $\frac{d^2T}{dy^2}=0$ و $\frac{d^2T}{dz^2}=0$

۳- انتقال حرارت پایاست بنابراین: $\frac{dT}{dt}=0$

$$\Longrightarrow \frac{d^2T}{dx^2}-\frac{hP}{kA}(T-T_\infty)=0$$

مساله ۱۵-۳)

برای پره سمت راست: $\frac{d^2\theta_1}{dx^2}-\frac{hP}{k_1A}\theta_1=0$

$$BC\begin{cases} x\longrightarrow\infty: & T_1=T_\infty,\ \theta_1(x)=0\quad (1) \\ x=0: & q''=-k_1\frac{d\theta_1}{dx}-k_2\frac{d\theta_2}{dx}\quad (2)\end{cases},\ \theta_1=T_1-T_\infty$$

برای پره سمت چپ: $\frac{d^2\theta_2}{dx^2} - \frac{hP}{k_2A}\theta_2 = 0$

$$BC\begin{cases} x \to -\infty: & T_2 = T_\infty\,, \theta_2(x) = 0 \quad (3) \\ x = 0: & T_1 = T_2\,, \theta_1 = \theta_2 \quad (4) \end{cases} \quad ,\theta_2 = T_2 - T_\infty$$

$$m^2 = \frac{hP}{kA} \quad \theta_1 = Ae^{-m_1x} + Be^{m_1x} \xrightarrow{(1)} B = 0 \quad , \quad \theta_1(x) = Ae^{-m_1x}$$

$$\theta_2 = Ce^{-m_2x} + De^{m_2x} \xrightarrow{(3)} C = 0 \quad , \quad \theta_2(x) = De^{m_2x}$$

$$(4) \to \theta_1(0) = \theta_2(0) \Rightarrow A = D$$

$$(2) \to k_1Am_1e^{-m_1x} - k_2Dm_2e^{m_2x} = q'' \Rightarrow k_1Am_1 - k_2Am_2 = q''$$

$$\Rightarrow A = \frac{q''}{k_1m_1 - k_2m_2}$$

$$\Rightarrow \begin{cases} \theta_1(x) = \frac{q''}{k_1m_1 - k_2m_2}e^{-m_1x} \\ \theta_2(x) = \frac{q''}{k_1m_1 - k_2m_2}e^{m_2x} \end{cases}$$

مساله ۱۸-۳)

فرض می‌کنیم که صفحه در جهت x متمرکز و در جهت y توزیع یافته است و ضخامت جهت سوم را W فرض نموده و از انتقال حرارت در این جهت صرف‌نظر می‌کنیم.

$$q_y.A\Big|_y - q_y.A\Big|_{y+dy} - 2h_3.dy.w(T - T_\infty) = 0$$

$$-\frac{d}{dy}\left(-k2Lw\frac{dT}{dy}\right)dy - 2h_3.dy.w(T - T_\infty) = 0$$

$$\Rightarrow \frac{d^2T}{dy^2} - \frac{h_3w}{kL}(T - T_\infty) = 0$$

به دلیل اینکه $\delta \gg 2L$ از عبارت دوم صرف‌نظر می‌کنیم

$$\frac{d^2T}{dy^2} = 0 \Rightarrow T = c_1y + c_2 \qquad \theta = T - T_\infty \Rightarrow \theta = c_1y + c_2$$

$$BC\begin{cases} y = 0: -k\frac{dT}{dy} = q'' + h_1(T - T_\infty) \\ \qquad\qquad \Rightarrow -k\frac{d\theta}{dy} = q'' + h_1\theta \Rightarrow -kc_1 - h_1c_1 = q'' \\ y = \delta: -k\frac{dT}{dy} = h_2(T - T_\infty) \Rightarrow -k\frac{d\theta}{dy} = h_2\theta \Rightarrow -kc_1 = h_2(c_1\delta + c_2) \end{cases}$$

$$\Rightarrow \begin{cases} (-kc_1 - h_1c_2 = q'') \times \frac{h_2}{h_1} \\ (k + h_2\delta)c_1 + h_2c_2 = 0 \end{cases} \Rightarrow \begin{cases} -k\frac{h_2}{h_1}c_1 - h_2c_2 = \frac{q''h_2}{h_1} \\ (k + h_2\delta)c_1 + h_2c_2 = 0 \end{cases} \Rightarrow$$

$$\left(-\frac{h_2}{h_1}k + k + h_2\delta\right)c_1 = \frac{q''h_2}{h_1}$$

$$\Rightarrow c_1 = \frac{q''h_2}{k(h_1 - h_2) + h_1h_2\delta} \quad , \quad c_2 = \frac{-kq''h_2}{h_1[k(h_1 - h_2) + h_1h_2\delta]} - \frac{q''}{h_1}$$

$$\Rightarrow \theta = T - T_\infty = \frac{q''h_2y}{k(h_1 - h_2) + h_1h_2\delta} - \frac{kq''h_2}{h_1[k(h_1 - h_2) + h_1h_2\delta]} - \frac{q''}{h_1}$$

$$\Rightarrow \theta = T - T_\infty = \frac{q''h_2}{k(h_1 - h_2) + h_1h_2\delta}\left[y - \frac{k}{h_1}\right] - \frac{q''}{h_1}$$

مساله ۱۹-۳)

برای بخشی که تولید انرژی دارد (u'''):

$$q_x.A - q_{x+dx}.A + u'''.A.dx - h.P.dx.(T_1 - T_\infty) = 0$$

$$-A.\frac{dq_x}{dx}.dx + u'''.A.dx - h.P.dx.(T_1 - T_\infty) = 0$$

$$q_x = -k\frac{dT_1}{dx} \Rightarrow \frac{d^2T_1}{dx^2} - \frac{hP}{kA}(T_1 - T_\infty) + \frac{u'''}{k} = 0 \qquad -L < x < L$$

برای بخشی که تولید انرژی ندارد:

$$q_x.A - q_{x+dx}.A - h.P.dx(T_2 - T_\infty) = 0$$

$$-\frac{dq_x}{dx}.A.dx - h.P.dx.(T_2 - T_\infty) = 0$$

$q_x = -k\frac{dT_2}{dx} \Rightarrow \frac{d^2T_2}{dx^2} - \frac{hP}{kA}(T_2 - T_\infty) = 0$ برای $x > L$ و $x < -L$

فرض $\theta_1 = T_1 - T_\infty$ و $\theta_2 = T_2 - T_\infty$

برای $-L < x < L \Rightarrow \frac{d^2\theta_1}{dx^2} - \frac{hP}{kA}\theta_1 + \frac{u'''}{k} = 0$

برای $x > L$ و $x < -L \Rightarrow \frac{d^2\theta_2}{dx^2} - \frac{hP}{kA}\theta_2 = 0$

$$BC\begin{cases} \frac{dT_1(0)}{dx} = 0 \Rightarrow \frac{d\theta_1(0)}{dx} = 0 \\ T_1(L) = T_2(L) \Rightarrow \theta_1(L) = \theta_2(L) \\ \frac{dT_1(L)}{dx} = \frac{dT_2(L)}{dx} = 0 \Rightarrow \frac{d\theta_1(L)}{dx} = \frac{d\theta_2(L)}{dx} \\ T_1(\infty) = T_\infty \Rightarrow \theta_2(\infty) = 0 \end{cases}$$

معادله اول:

$$\frac{d^2\theta_1}{dx^2} - \frac{hP}{kA}\theta_1 + \frac{u'''}{k} = 0 \quad m^2 = \frac{hP}{kA} \Rightarrow \frac{d^2\theta_1}{dx^2} - m^2\theta_1 + \frac{u'''}{k} = 0$$

$$\Rightarrow \frac{d^2\theta_1}{dx^2} - m^2\theta_1 = -\frac{u'''}{k} \Rightarrow \theta_1 = C_1e^{mx} + C_2e^{-mx} + \frac{u'''}{km^2}$$

معادله دوم:

$$\frac{d^2\theta_2}{dx^2} - \frac{hP}{kA}\theta_2 = 0 \quad m^2 = \frac{hP}{kA} \Rightarrow \frac{d^2\theta_2}{dx^2} - m^2\theta_2 = 0$$

$$\Rightarrow \theta_2 = D_1e^{mx} + D_2e^{-mx}$$

با استفاده از شرایط مرزی: $\frac{d\theta_1}{dx}(0) = 0 \Rightarrow C_1 - C_2 = 0 \text{ or } C_1 = C_2$

$$\theta_2(x \to \infty) \to 0 \Rightarrow D_1 = 0$$

$$\theta_1(L) = \theta_2(L) \Rightarrow C_1e^{mL} + C_2e^{-mL} + \frac{u'''}{km^2} = D_2e^{-mL}$$

$$\Rightarrow D_2 = C_1(1 + e^{2mL}) + \frac{u'''}{km^2}e^{mL}$$

$$\frac{d\theta_1(L)}{d_x} = \frac{d\theta_2(L)}{d_x} \Rightarrow C_1me^{mL} - C_2me^{-mL} = -mD_2e^{-mL}$$

$$\Rightarrow C_1(e^{mL} + e^{-mL}) = -D_2e^{-mL} \quad (i)$$

$$C_1(e^{mL} - e^{-mL}) = D_2e^{-mL} - \frac{u'''}{km^2} \quad (ii)$$

$$(i) + (ii) \Rightarrow 2C_1e^{mL} = -\frac{u'''}{km^2} \Rightarrow C_1 = -\frac{u'''}{2km^2}e^{-mL}$$

$$\Rightarrow D_2 = \frac{u'''}{2km^2}(e^{mL} - e^{-mL}) = \frac{u'''}{km^2}\sinh(mL)$$

سپس: $\theta_1(x) = -\frac{u'''}{2km^2}e^{mx}.e^{-mL} - \frac{u'''}{2km^2}e^{-mx}.e^{-mL} + \frac{u'''}{km^2}$

$$\Rightarrow T_1(x) = -\frac{u'''}{2km^2}\left[e^{m(x-L)} - e^{-m(x+L)}\right] + \frac{u'''}{km^2} + T_\infty, \ -L < x < L$$

$$, m = \sqrt{\frac{hP}{kA}}$$

و

$$\theta_2(x) = \frac{u'''}{km^2}\sinh(mL)\, e^{-mx} \Rightarrow T_2(x) = \frac{u'''A}{hP}\sinh\left(\sqrt{\frac{hP}{kA}}L\right).e^{-\sqrt{\frac{hP}{kA}}x} + T_\infty$$

مساله ۲۰-۳)

برای بخش درون چای:

$$q_x = -k\frac{dT_1}{dx}, q_x.A - q_{x+dx}.A - h_0.P.dx.(T_0 - T_1) = 0$$

$$\Rightarrow -\frac{dq_x}{dx}.dx.A - h.P.dx.(T_1 - T_0) = 0 \Rightarrow \frac{d^2T_1}{dx^2} - \frac{h_0P}{kA}(T_1 - T_0) = 0$$

برای بخش خارج از چای:

$$q_x = -k\frac{dT_2}{dx}, q_x.A - q_{x+dx}.A - h.P.dx.(T_2 - T_\infty) = 0$$

$$\Rightarrow -\frac{dq_x}{dx}.dx.A - h.P.dx(T_2 - T_\infty) = 0 \Rightarrow \frac{d^2T_2}{dx^2} - \frac{hP}{kA}(T_2 - T_\infty) = 0$$

فرض $\theta_1 = T_1 - T_0$ و $\theta_2 = T_2 - T_\infty$

معادله اول: $\frac{d^2\theta_1}{dx^2} - \frac{h_0P}{kA}\theta_1 = 0 \qquad 0 < x < L \qquad m_0^2 = \frac{h_0P}{kA}$

معادله دوم: $\frac{d^2\theta_2}{dx^2} - \frac{hP}{kA}\theta_2 = 0 \qquad L < x < 2L \quad m^2 = \frac{hP}{kA}$

$$BC\begin{cases}\frac{dT_1(0)}{dx}=0\Rightarrow\frac{d\theta_1(0)}{dx}=0\\ \frac{dT_2(2L)}{dx}=0\Rightarrow\frac{d\theta_2(2L)}{dx}=0\\ T_1(L)=T_2(L)\Rightarrow T_1(L)-T_0+T_0=T_2(L)-T_\infty+T_\infty\\ \Rightarrow\theta_1(L)+T_0=\theta_2(L)+T_\infty\Rightarrow\theta_1(L)-\theta_2(L)=T_\infty-T_0\\ \frac{dT_2(L)}{dx}=\frac{dT_2(L)}{dx}\Rightarrow\frac{d\theta_1(L)}{dx}=\frac{d\theta_2(L)}{dx}\end{cases}$$

$\theta_1(x)=C_1\sinh(m_0x)+C_2\cosh(m_0x),$

$\theta_2(x)=D_1\sinh(mx)+D_2\cosh(mx)$

$\frac{d\theta_1(0)}{dx}=0\Rightarrow C_1m_0\cosh(m_00)+C_2m_0\sinh(m_00)=0\Rightarrow C_1=0$

$\frac{d\theta_2(2L)}{dx}=0\Rightarrow D_1m\cosh(m2L)+D_2m\sinh(m2L)=0$

$\Rightarrow D_1=-D_2\tanh(m2L)$

$\frac{d\theta_1(L)}{dx}=\frac{d\theta_2(L)}{dx}\Rightarrow C_2m_0\sinh(m_0L)=D_1m\cosh(mL)+D_2m\sinh(mL)$

$C_2m_0\sinh(m_0L)=D_2(m\sinh(mL)-m\cosh(mL)\tanh(m2L))$

$\Rightarrow D_2=\frac{C_2m_0\sinh(m_0L)}{m\sinh(mL)-m\cosh(mL)\tanh(m2L)}$

$\theta_1(L)-\theta_2(L)=T_\infty-T_0$

$\Rightarrow C_2\cosh(m_0L)-D_1\sinh(mL)-D_2\cosh(mL)=T_\infty-T_0$

$\Rightarrow C_2\cosh(m_0L)+D_2(\tanh(m2L)\,.\sinh(mL)-\cosh(mL))=T_\infty-T_0$

$\Rightarrow$

$C_2\cosh(m_0L)+\frac{C_2m_0\sinh(m_0L)}{m\sinh(mL)-m\cosh(mL)\tanh(m2L)}(\tanh(m2L)\,.\sinh(mL)$

$\cosh(mL))=T_\infty-T_0$

$\Rightarrow C_2=$

$$\frac{(T_\infty-T_0)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))}{\cosh(m_0L)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))+m_0\sinh(m_0L)(\tanh(m2L).\sinh(mL)-\cosh(mL))}$$

$\Rightarrow D_2=$

$$\frac{(T_\infty-T_0)m_0\sinh(m_0L)}{\cosh(m_0L)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))+m_0\sinh(m_0L)(\tanh(m2L).\sinh(mL)-\cosh(mL))}$$

$\Rightarrow D_1=$

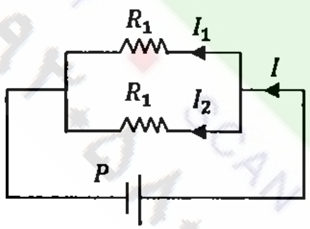
$$\frac{-(T_\infty-T_0)m_0\sinh(m_0L).\tanh(m2L)}{\cosh(m_0L)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))+m_0\sinh(m_0L)(\tanh(m2L).\sinh(mL)-\cosh(mL))}$$

$\Rightarrow\theta_1(x)=T_1(x)-T_0=$

$$\frac{-(T_\infty-T_0)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))\cosh(m_0x)}{\cosh(m_0L)(m\sinh(mL)-m\cosh(mL)\tanh(m2L))+m_0\sinh(m_0L)(\tanh(m2L).\sinh(mL)-\cosh(mL))}$$

$\Rightarrow\theta_2(x)=T_2(x)-T_\infty=D_1\sinh(mx)+D_2\cosh(mx)$

مساله ۲۱-۳)



$R_{eq}=\frac{R_1R_2}{R_1+R_2}$, $P=R_{eq}.I^2=\frac{R_1R_2}{R_1+R_2}I^2$

$\Rightarrow I=\sqrt{\frac{P(R_1+R_2)}{R_1R_2}}$ $\quad\left.\begin{matrix}I_1R_1=I_2R_2\\ I_1+I_2=I\end{matrix}\right\}\Rightarrow I_2=\frac{R_2}{R_1}I_1\ ,\ I_2\left(\frac{R_2}{R_1}+1\right)=$

$\sqrt{\frac{P(R_1+R_2)}{R_1R_2}}$

$\Rightarrow I_1\left(\frac{R_1+R_2}{R_2}\right)=\sqrt{\frac{p(R_1+R_2)}{R_1R_2}}\Rightarrow I_1=\sqrt{\frac{PR_1}{R_1(R_1+R_2)}}\ ,\ I_2=\sqrt{\frac{PR_1}{R_2(R_1+R_2)}}$

با فرض اینکه هر ظلع مربع L می‌باشد، انرژی تولیدی با تقسیم نمودن بر واحد حجم به دست می‌آید:

$u'''_1=\frac{R_1I_1^2}{2AL}=\frac{\frac{PR_1R_2}{R_1(R_1+R_2)}}{2AL}=\frac{PR_2}{2AL(R_1+R_2)}$ انرژی تولید شده در واحد حجم

$u'''_2=\frac{R_2I_2^2}{2AL}=\frac{\frac{PR_1R_2}{R_2(R_1+R_2)}}{2AL}=\frac{PR_1}{2AL(R_1+R_2)}$

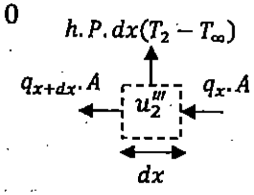
به دلیل تقارن شکل نصف آن را برای انجام تحلیل در نظر می‌گیریم.

یک المان با ابعاد dx و dy به ترتیب در جهت‌های x و y در نظر می‌گیریم

در جهت x :

$$q_x.A - q_{x+dx}.A + u'''_2.A.dx - h.P.dx(T_2 - T_\infty) = 0$$

$$k_2A\frac{d^2T_2}{dx^2}dx + u'''_2A.dx - hPdx(T_2 - T_\infty) = 0$$



$$\Longrightarrow \frac{d^2T_2}{dx^2} + \frac{u'''_2}{k_2} - \frac{hP}{k_2A}(T_2 - T_\infty) = 0$$

$$T_2 - T_\infty = \theta_2 \Longrightarrow \frac{d^2\theta_2}{dx^2} - m_2^2\theta_2 + \frac{u'''_2}{k_2} = 0$$

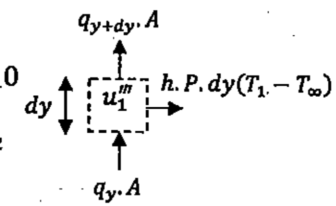
$$\Longrightarrow \theta_2(x) = C_3\cosh(m_2x) + C_4\sinh(m_2x) + \frac{u'''_2}{km_2}$$

در جهت y :

$$q_y.A - q_{y+dy}.A + u'''_1.A.dy - h.P.dy.(T_1 - T_\infty) = 0$$

$$\Longrightarrow \frac{d^2T_1}{dx^2} - \frac{hP}{k_1A}(T_1 - T_\infty) + \frac{u'''_1}{k_1} = 0$$

$$,\quad T_1 - T_\infty = \theta_1 \Longrightarrow \frac{d^2\theta_1}{dx^2} - m_1^2\theta_1 + \frac{u'''_1}{k_1} = 0$$



$$BC\begin{cases} x = 0\,, y = 0 \Longrightarrow T_1 = T_2 \Longrightarrow \theta_1 = \theta_2 \\ x = 0\,, y = 0 \Longrightarrow \frac{d\theta_1}{dy} = \frac{d\theta_2}{dx} \\ x = L \Longrightarrow \frac{d\theta_2}{dx} = 0 \\ y = L \Longrightarrow \frac{d\theta_1}{dy} = 0 \end{cases}$$

$$\theta_1(y) = C_1\cosh(m_1y) + C_2\sinh(m_1y) + \frac{u'''_1}{k_1m_1^2}$$

$$\theta_2(x) = C_3\cosh(m_2x) + C_4\sinh(m_2x) + \frac{u'''_2}{km_2}$$

$$x = 0\,, y = 0 \Longrightarrow C_1 + \frac{u'''_1}{k_2m_2^2} = C_3 + \frac{u'''_2}{k_2m_2^2} \Longrightarrow C_1 - C_3 = \frac{u'''_2}{k_2m_2^2} - \frac{u'''_1}{k_1m_1^2}$$

$$\frac{d\theta_1}{dy} = C_1m_1\sinh(m_1y) + C_2m_1\cosh(m_1y)$$

$$\frac{d\theta_2}{dx} = C_3m_2\sinh(m_2x) + C_4m_2\cosh(m_2x)$$

$$C_2m_1 = C_4m_2$$

$$C_3m_2\sinh(m_2L) + C_4m_2\cosh(m_2L) = 0 \Longrightarrow C_4 = -C_3\tanh(m_2L)$$

$$C_1m_1\sinh(m_1L) + C_2m_1\cosh(m_1L) = 0 \Longrightarrow C_2 = -C_1\tanh(m_1L)$$

$$\Longrightarrow C_1m_1\tanh(m_1L) = C_3m_2\tanh(m_2L) \Longrightarrow C_3 = \frac{C_1m_1\tanh(m_1L)}{m_2\tanh(m_2L)}$$

$$C_1 - C_3 = \frac{u'''_2}{k_2m_2^2} - \frac{u'''_1}{k_1m_1^2} \Longrightarrow C_1 - \frac{C_1m_1\tanh(m_1L)}{m_2\tanh(m_2L)} = \frac{u'''_2}{k_2m_2^2} - \frac{u'''_1}{k_1m_1^2}$$

$$\Longrightarrow C_1\left(1 - \frac{m_1\tanh(m_1L)}{m_2\tanh(m_2L)}\right) = \frac{k_1m_1^2u'''_2 - k_2m_2^2u'''_1}{k_1k_2m_1^2m_2^2}$$

$$\Longrightarrow C_1 = \frac{m_2\tanh(m_2L)(k_1m_1^2u'''_2 - k_2m_2^2u'''_1)}{k_1k_2m_1^2m_2^2(m_2\tanh(m_2L) - m_1\tanh(m_1L))}$$

$$\Longrightarrow C_3 = \frac{m_1 tanh m_1 L(k_1m_1^2u'''_2 - k_2m_2^2u'''_1)}{k_1k_2m_1^2m_2^2(m_2\tanh(m_2L) - m_1\tanh(m_1L))}$$

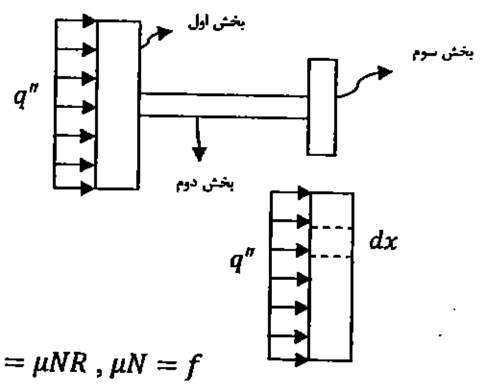
$$C_4 = \frac{-m_1\tanh(m_1L).\tanh(m_2L)(k_1m_1^2u'''_2 - k_2m_2^2u'''_1)}{k_1k_2m_1^2m_2^2(m_2\tanh(m_2L) - m_1\tanh(m_1L))}$$

$$C_2 = \frac{-m_1.m_2.\tanh(m_1L).\tanh(m_2L).(k_1m_1^2u'''_2 - k_2m_2^2u'''_1)}{k_1k_2m_1^2m_2^2(m_2\tanh(m_2L) - m_1\tanh(m_1L))}$$

مساله ۲۲-۳)

مساله را به سه بخش تقسیم می‌کنیم.

با استفاده از موازنه مومنتم: مومنتم به دلیل نیروی P =مومنتم به دلیل نیروی اصطکاک



$$PL = \mu NR\,, \mu N = f$$

با استفاده از قانون اول ترمودینامیک: کار حاصل از نیروی اصطکاک = گرمای تولید شده

$$\Longrightarrow Q = f.R.\omega = \frac{PL}{R}.R.\omega = P.L.\omega$$

$$\Longrightarrow \frac{Q}{A} = q'' = \frac{PL\omega}{2\pi RL}$$

برای بخش اول:

$$A = 2\pi R\delta_1 \,,\, q^{''} = \frac{pLw}{2\pi RL} \,,\, q_x = -k\frac{dT_1}{d_x}$$

$$q_x.A - q_{x+dx}.A + q^{''}(2\pi R)dx - 2\pi h(R-\delta_1)(T_1 - T_\infty)dx = 0$$

$$q^{''}(2\pi R)dx = \frac{dq_x}{dx}dx.A + 2\pi h(R-\delta_1)(T_1 - T_\infty)dx$$

$$2\pi Rq^{''} = -kA\frac{d^2T_1}{dx^2} + 2\pi h(R-\delta_1)(T_1 - T_\infty)$$

$$\Longrightarrow 2\pi R\frac{PL\omega}{2\pi Rl} = -k2\pi R\delta_1\frac{d^2T_1}{dx^2} + 2\pi h(R-\delta_1)(T_1 - T_\infty)$$

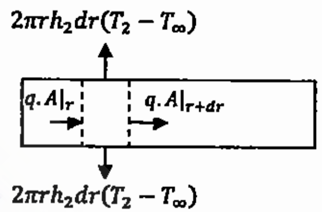
$$\Longrightarrow \frac{d^2T_1}{dx^2} - \frac{h(R-\delta_1)}{hR\delta_1}(T_1 - T_\infty) + \frac{Pl\omega}{2\pi lRk\delta_1} = 0 \qquad m_1^2 = \frac{h(R-\delta)}{kR\delta_1}$$

$$\theta_1 = T_1 - T_\infty \Longrightarrow \frac{d^2\theta_1}{dx^2} - m_1^2\theta_1 = \frac{-Pl\omega}{2\pi Rlk\delta_1}$$

برای بخش دوم:

$$q.A|_r - q.A|_{r+dr} = 2(2\pi rh_2dr)(T_2 - T_\infty)$$

$$\Longrightarrow \frac{d}{dr}\left(Ak\frac{dT_2}{dr}\right) - 4\pi h_2r(T_2 - T_\infty) = 0$$



$$2\pi k\delta_2\frac{d}{dr}\left(r\frac{dT_2}{dr}\right) - 4\pi h_2r(T_2 - T_\infty) = 0$$

$$\frac{d}{dr}\left(r\frac{dT_2}{dr}\right) - \frac{2h_2}{k\delta_2}r(T_2 - T_\infty) = 0$$

$$\theta_2(r) = T_2 - T_\infty \Longrightarrow \frac{d}{dr}\left(r\frac{d\theta_2}{dr}\right) - \frac{2h_2}{k\delta_2}r\theta_2 = 0$$

$$m_2^2 = \frac{2h}{k\delta_2}$$

$$\Longrightarrow \frac{d}{dr}\left(r\frac{d\theta_2}{dr}\right) - m_2^2r\theta_2 = 0$$

برای بخش سوم:

با توجه به فرض $\theta_3 = T_3 - T_\infty$ و $T_3 - T_\infty \Longrightarrow \theta_3 = 0$

$$BC\begin{cases} (1) & \frac{d\theta_1(L/2)}{dx} = 0 \\ (2) & \theta_1(0) = \theta_2(R) \\ (3) & 2\times 2\pi R\delta_1k\frac{d\theta_1(0)}{dx} = 2\pi R\delta_2k\frac{d\theta_2(R)}{dr} \Longrightarrow 2\delta_1\frac{d\theta_1(0)}{dx} = \delta_2\frac{d\theta_2(R)}{dr} \\ (4) & \theta_2(R') = 0 \end{cases}$$

$$\frac{d^2\theta_1}{dx^2} - m_1^2\theta_1 = \frac{-Pl\omega}{2\pi Rlk\delta_1} \Longrightarrow \theta_1(\mathrm{x}) = \mathrm{A}e^{\mathrm{m_1x}} + \mathrm{B}e^{-\mathrm{m_1x}} + \frac{Pl\omega}{2\pi Rlk\delta_1m_1^2}$$

$$\frac{d}{dr}\left(r\frac{d\theta_2}{dr}\right) - m_2^2r\theta_2 = 0 \Longrightarrow \text{تابع بسل} \Longrightarrow \alpha = 1\,,\beta = 1\,,\gamma = \pm\mathrm{im}_2$$

$$\Longrightarrow \beta - \alpha + 2 = 2 \neq 0, v = \frac{1-\alpha}{\beta-\alpha+2} = 0\ \mu = \frac{2}{\beta-\alpha+2} = 1 \Longrightarrow \frac{v}{\mu} = \frac{1-\alpha}{2} = 0$$

$$\Longrightarrow \theta_2(r) = c_1I_0(m_2r) + Dk_0(m_2r) \qquad m_2^2 = \frac{2h}{k\delta_2}$$

$$(1) \Longrightarrow \frac{d\theta_1(L/2)}{dx} = 0 \Longrightarrow \mathrm{A}m_1e^{-\frac{m_1l}{2}} = 0 \Longrightarrow \mathrm{B} = \mathrm{A}e^{\mathrm{m_1l}}$$

$$(2) \Longrightarrow A + B + \frac{Pl\omega}{2\pi Rlk\delta_1m_1^2} = CI_0(m_2R) + Dk_0(m_2R)$$

$$(3) \Longrightarrow 2\delta_1m_1\,(A-B) = \delta_2m_2\,(m_2\,R) - CI_1(m_2R)$$

$$(4) \Longrightarrow C_1\ I_0(m_2R') + Dk_0(m_2R') = 0$$

$$\Longrightarrow D = \frac{Pl\omega I_0(m_2R')}{\pi Rlkm_1\,(\delta_2m_2k_1(m_2R)I_0(m_2R') - 4\delta_2m_2I_1(m_2R)k_0(m_2R'))}$$

$$C = \frac{-Pl\omega k_0(m_2R')}{\pi Rlkm_1\,(\delta_2m_2k_1(m_2R)I_0(m_2R') - 4\delta_2m_2I_1(m_2R)k_0(m_2R'))}$$

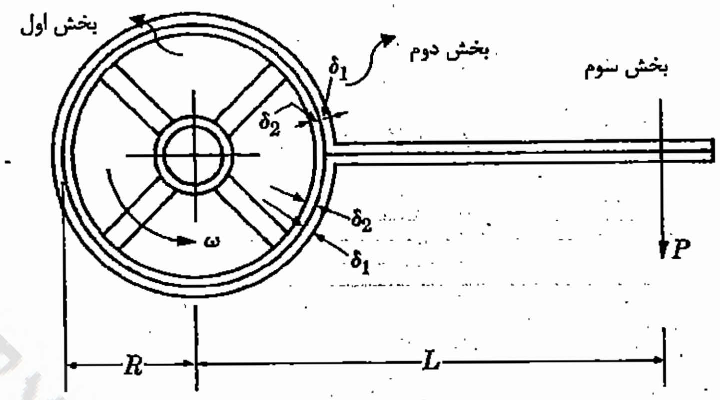
$$B =$$

$$\frac{Pl\omega I_0(m_2R')(\delta_1m_1[k_0(m_2R)I_0(m_2R') - k_0(m_2R')I_0(m_2R)])}{\pi Rlkm_1\,(\delta_2m_2k_1(m_2R)I_0(m_2R') - 4\delta_2m_2I_1(m_2R)k_0(m_2R'))} + \frac{(-3\delta_2m_2I_1(m_2R)k_0(m_2R'))}{\delta_1m_1I_0(m_2R')}$$

$$A =$$

$$\frac{Pl\omega I_0(m_2R')(\delta_2m_2k_1(m_2R)I_0(m_2R') + \delta_1m_1k_0(m_2R)I_0(m_2R'))}{2\pi Rl\delta_1m_1^2k(\delta_2m_2k_1(m_2R)I_0(m_2R') - 4\delta_2m_2I_1(m_2R)k_0(m_2R'))} - \frac{(-\delta_1m_1k_0(m_2R')I_0(m_2R) - \delta_2m_2I_1(m_2R)k_0(m_2R'))}{2\pi Rl\delta_1m_1^2k(\delta_2m_2k_1(m_2R)I_0(m_2R') - 4\delta_2m_2I_1(m_2R)k_0(m_2R'))}$$

مساله ۲۳-۳)



با استفاده از موازنه مومنتم: $\Rightarrow PL = \mu N.R = f.R$

قانون اول ترمودینامیک: گرمای تولیدی $= Q =$ کار حاصل از نیروی اصطکاک $= R\omega f =$

$\frac{PL}{R}R\omega = Pl\omega$

با فرض اینکه $\delta_1 < \delta_2$ شار حرارت $= q'' = \frac{Pl\omega}{2\pi R\delta_1}$

مساله را به سه بخش تقسیم می‌کنیم:

برای بخش اول:

به دلیل تقارن $\pi/4$ ربع اول را در نظر می‌گیریم

$$q''R\delta_1 d\theta + q_\theta \delta_2 \delta_1 = h(R-\delta_2)d\theta.\delta_1(T_1 - T_\infty) + q_{\theta+d\theta}.\delta_1\delta_2$$

$$\Psi = T_1 - T_\infty \Rightarrow -k\frac{d\Psi^2}{d\theta^2}\delta_1 d\theta + h(R-\delta_2)\Psi d\theta = q''Rd\theta$$

$$\Rightarrow \frac{d^2\Psi}{d\theta^2} - \frac{Rh(R-\delta_2)}{k}\Psi + \frac{q''R^2}{k} = 0$$

برای بخش دوم: δ_1 را به عنوان عمق و δ_3 را به عنوان ضخامت آن در نظر می‌گیریم.

$$q_{\text{convection}} = [2h\delta_1(T_2 - T_\infty) + 2h\delta_3(T_2 - T_\infty)]dx$$

$$q_x.A = q_{x+dx}.A + q_{\text{conv}} \Rightarrow \frac{dq_x}{dx}\delta_1\delta_3 + 2h(\delta_1 + \delta_3)(T_2 - T_\infty) = 0$$

$$\Rightarrow q_x = -k\frac{d^2T_2}{dx} \Rightarrow \frac{d^2T_2}{dx^2} - \frac{2h(\delta_1+\delta_3)}{k\delta_1\delta_3}(T_2 - T_\infty) = 0$$

$$\Phi = T_2 - T_\infty \Rightarrow \frac{d^2\theta}{dx^2} - \frac{2h(\delta_1+\delta_3)}{k\delta_1\delta_3}\Phi = 0$$

برای بخش سوم:

$$T_3 - T_\infty \Rightarrow \theta_3 = 0$$

$$BC\begin{cases} (1) \quad \frac{dT_1(0)}{d\theta} = 0 \Rightarrow \frac{d\Psi(0)}{d\theta} = 0 \\ (2) \quad T_1\left(\frac{\pi}{A}\right) = T_2(R) \Rightarrow \Psi\left(\frac{\pi}{A}\right) = \Phi(R) \\ (3) \quad T_2(R_i) = T_\infty \Rightarrow \Phi(R_i) = 0 \\ (4) \quad 2\times 2\pi R\delta_1 \frac{dT_1\left(\frac{\pi}{A}\right)}{d\theta} = \delta_3\delta_1\frac{dT_2(R)}{dx} \Rightarrow 4\pi R\frac{d\Psi\left(\frac{\pi}{A}\right)}{d\theta} = \delta_3\frac{d\phi(R)}{dx} \end{cases}$$

$$\frac{d^2\Psi}{d\theta^2} - \frac{Rh(R-\delta_2)}{k}\Psi + \frac{q''R^2}{k} = 0 \qquad \frac{Rh(R-\delta_2)}{k} = m_1^2$$

$$\Psi = A\sinh m_1\theta + B\cosh m_1\theta + \frac{q''R^2}{2km_1^2}$$

$$\frac{d^2\phi}{dx^2} - \frac{2h(\delta_1+\delta_3)}{k\delta_1\delta_3}\phi = 0 \qquad m_2^2 = \frac{2h(\delta_1+\delta_3)}{k\delta_1\delta_3}$$

$$\phi = c\sinh m_2x + B\cosh m_2x$$

$$(1) \Rightarrow \frac{d\Psi(0)}{d\theta} = 0 \Rightarrow A = 0$$

$$(2) \Rightarrow \Psi\left(\frac{\pi}{A}\right) = \phi(R) \Rightarrow B\cosh m_1\frac{\pi}{4} + \frac{q''R^2}{2km_1^2} = c\sinh m_2R + D\cosh m_2R$$

$$(3) \Rightarrow \phi(R_i) = 0 \Rightarrow c\sinh m_2R_i + D\cosh m_2R_i = 0$$

$$(4) \Rightarrow 4\pi R\frac{d\Psi\left(\frac{\pi}{4}\right)}{d\theta} = \delta_3\frac{d\phi(R)}{dx}$$

$$\Rightarrow 4\pi R\left(Am_1\cosh m_1\frac{\pi}{4} + Bm_1\sinh m_1\frac{\pi}{4}\right) = \delta_3(Cm_2\cosh m_2R + Dm_2\sinh m_2R)$$

$$(3) \Rightarrow D = -C\tanh m_2R_i$$

$$(2) \Rightarrow B\cosh m_1\frac{\pi}{4} + \frac{q''R^2}{2km_1^2} = C(\sinh m_2R - \tanh m_2R_i\cosh m_2R)$$

$$\Rightarrow \mathrm{B} = -\frac{q''R^2}{2km_1^2\cosh m_1\frac{\pi}{4}} + \frac{\sinh m_2R - \tanh m_2R_i\cosh m_2R}{\cosh m_1\frac{\pi}{4}}C$$

$$(4) \Rightarrow 4\pi Rm_1\sinh m_1\frac{\pi}{4}\left(-\frac{q''R^2}{2km_1^2\cosh m_1\frac{\pi}{4}} + C\frac{\sinh m_2R - \tanh m_2R_i\cosh m_2R}{\cosh m_1\frac{\pi}{4}}\right)$$

$$= \delta_3(Cm_2\cosh m_2R - Cm_2\tanh m_2R_i\sinh m_2R)$$

$$4\pi Rm_1\sinh m_1\frac{\pi}{4}.\frac{q''R^2}{2km_1^2\cosh m_1\frac{\pi}{4}} = C\Big[4\pi Rm_1\tanh m_1\frac{\pi}{4}(\sinh m_2R -$$

$$\tanh m_2R_i.\cosh m_2R) - \delta_3m_2\cosh m_2R + m_2\delta_3\tanh m_2R_i.\sinh m_2R\Big]$$

$$\Rightarrow \mathrm{C} =$$

$$\frac{2\pi R^3 + \cosh m_1\frac{\pi}{4}q''}{k_1m_1\left[4\pi Rm_1\tanh m_1\frac{\pi}{4}(\sinh m_2R - \tanh m_2R_i.\cosh m_2R) - \delta_3m_2\cosh m_2R + m_2\delta_3\tanh m_2R_i.\sinh m_2R\right]}$$

$$\Rightarrow \mathrm{D} = -\mathrm{C}\tanh m_2R_i =$$

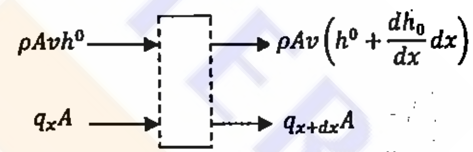
$$\frac{-2\pi R^3\tanh m_1\frac{\pi}{4}q''.\tanh m_2R_i}{k_1m_1\left[4\pi Rm_1\tanh m_1\frac{\pi}{4}(\sinh m_2R - \tanh m_2R_i.\cosh m_2R) - \delta_3m_2\cosh m_2R + m_2\delta_3\tanh m_2R_i.\sinh m_2R\right]}$$

$$\Rightarrow \Psi \checkmark \;,\; \phi \checkmark$$

مساله ۲۴-۳)

برای بخش $x < 0$:

$$q_x = -k\frac{dT_1}{dx}, h^0 = cT_1$$

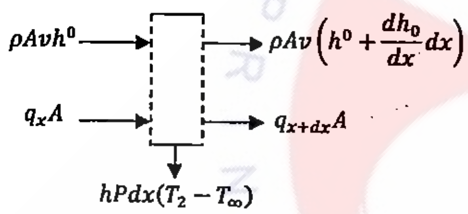


$$\rho Avh^0 + q_xA = \rho Avh^0 + \rho Av\frac{dh_0}{dx}dx + q_xA + \frac{dq_x}{dx}dx.A$$

$$\rho Av\frac{dh_0}{dx} + \frac{dq_x}{dx}A = 0 \Rightarrow \rho vAc\frac{dT_1}{dx} - kA\frac{d^2T_1}{dx^2} = 0 \Rightarrow \frac{d^2T_1}{dx^2} - \frac{\rho vc}{k}\frac{dT_1}{dx} = 0$$

$$T_1 - T_0 = \theta_1 \Rightarrow \frac{d^2\theta_1}{dx^2} - \frac{\rho vc}{k}\frac{d\theta_1}{dx} = 0$$

برای بخش $x > 0$:



$$q_x = -k\frac{dT_2}{dx}, h^0 = cT_2$$

$$q_xA + \rho Avh^0 = \rho vA\left(h^0 + \frac{dh_0}{dx}dx\right) + hPdx(T_2 - T_\infty) + q_{x+dx}A$$

$$0 = hPdx(T_2 - T_\infty) + \rho vA\frac{dh^0}{dx}dx + \frac{dq_x}{dx}dx.A$$

$$\Rightarrow \frac{d^2T_2}{dx^2} - \frac{\rho vc}{k}\frac{dT_2}{dx} - \frac{hP}{kA}(T_2 - T_\infty) = 0$$

$$T_2 - T_\infty = \theta_2 \Rightarrow \frac{d^2\theta_2}{dx^2} - \frac{\rho vc}{k}\frac{d\theta_2}{dx} - \frac{hP}{kA}\theta_2 = 0, \mathrm{x} > 0$$

$$BC\begin{cases} (1) & T_1(0) = T_2(0) \Rightarrow T_1(0) - T_0 + T_0 = T_2(0) - T_\infty + T_\infty \Rightarrow \\ & \theta_1(0) - \theta_2(0) = T_\infty - T_0 \\ (2) & \dfrac{dT_1(0)}{dx} = \dfrac{dT_2(0)}{dx} \Rightarrow \dfrac{d\theta_1(0)}{dx} = \dfrac{d\theta_2(0)}{dx} \\ (3) & T_1(x \to -\infty) = T_0 \Rightarrow \theta_1(-\infty) = 0 \\ (4) & T_2(x \to +\infty) = T_\infty \Rightarrow \theta_2(+\infty) = 0 \end{cases}$$

برای $\mathrm{x} < 0 \Rightarrow \frac{d^2\theta_1}{dx^2} - \frac{\rho vc}{k}\frac{d\theta_1}{dx} = 0, \frac{\rho vc}{k} = m^2 \Rightarrow \frac{d^2\theta_1}{dx^2} - m^2\frac{d\theta_1}{dx} = 0$

فرض: $\theta_1 = e^{\lambda x} \Rightarrow \lambda^2 - m^2\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = m^2$

$$\theta_1(x) = Ae^{\lambda_1x} + Be^{\lambda_2x} = A + Be^{m^2x}$$

برای $\mathrm{x} > 0 \Rightarrow \frac{d^2\theta_2}{dx^2} - \frac{\rho vc}{k}\frac{d\theta_2}{dx} = 0, n^2 = \frac{hP}{kA}, m^2 = \frac{\rho vc}{k}$

فرض: $\theta_2 = e^{\lambda x} \Rightarrow \lambda^2 - m^2\lambda - n^2 = 0 \Rightarrow \lambda = \frac{m^2 \pm \sqrt{m^4 + 4n^2}}{2}$

$$\theta_2(x) = Ce^{\frac{m^2+\sqrt{m^4+4n^2}}{2}x} + De^{\frac{m^2-\sqrt{m^4+4n^2}}{2}x}$$

$$(1) \Rightarrow (A + B) - (C - D) = \lambda_2D \Rightarrow D = \frac{m^2}{\lambda_2}B$$

$$(1)\ \Longrightarrow B-D=T_\infty-T_0 \Longrightarrow B\left(1-\frac{m^2}{\lambda_2}\right)=T_\infty-T_0$$

$$\Longrightarrow B=\frac{\lambda_2(T_\infty-T_0)}{\lambda_2-m^2},\ D=\frac{m^2(T_\infty-T_0)}{(\lambda_2-m^2)}$$

$$\Longrightarrow \theta_1(x)=T_1(x)-T_0=\frac{\lambda_2(T_\infty-T_0)}{\lambda_2-m^2}e^{m^2x}$$

$$\Longrightarrow \frac{T_1(x)-T_0}{T_\infty-T_0}=\frac{m^2+\sqrt{m^4+4n^2}}{-m^2+\sqrt{m^4+4n^2}}e^{m^2x}$$

$$\theta_2(x)=T_2(x)-T_\infty=\frac{m^2(T_\infty-T_0)}{(\lambda_2-m^2)}=e^{\left(\frac{m^2+\sqrt{m^4+4n^2}}{2}x\right)}$$

$$\Longrightarrow \frac{T_2(x)-T_\infty}{T_0-T_\infty}=\frac{2m^2}{-m^2+\sqrt{m^4+4n^2}}e^{\left(\frac{m^2+\sqrt{m^4+4n^2}}{2}x\right)}$$

مساله ۲۵-۳)

برای بخش $x<0$:

$$q_x=-k\frac{dT_1}{dx},\ h^0=cT_1$$

$$\rho Avh^0+q_xA=\rho vA\left(h^0+\frac{dh_0}{dx}dx\right)+q_xA+\frac{dq_x}{dx}dx.A$$

$$\Longrightarrow \rho vA\frac{dh^0}{dx}+\frac{dq_x}{dx}A=0 \Longrightarrow \rho vAc\frac{dT_1}{dx}-kA\frac{d^2T_1}{dx^2}=0$$

$$\Longrightarrow \frac{d^2T_1}{dx^2}-\frac{\rho vc}{k}\frac{dT_1}{dx}=0,\ \theta_1=T_1-T_0 \Longrightarrow \frac{d^2\theta_1}{dx^2}-\frac{\rho vc}{k}\theta_1=0$$

برای بخش $0<x<L$:

$$q_xA+\rho Avh^0=\rho vAh^0+\rho vA\frac{dh^0}{dx}dx+hPdx(T_2-T_\infty)+q_{x+dx}A$$

$$0=hPdx(T_2-T_\infty)+\rho vA\frac{dh^0}{dx}dx+\frac{dq_x}{dx}dx.A$$

$$\Longrightarrow \frac{d^2T_2}{dx^2}-\frac{\rho vc}{k}\frac{dT_2}{dx}-\frac{hP}{kA}(T_2-T_\infty)=0$$

$$\theta_2=T_2-T_\infty$$

$$\Longrightarrow \frac{d^2\theta_2}{dx^2}-\frac{\rho vc}{k}\frac{d\theta_2}{dx}-\frac{hP}{kA}\theta_2=0$$

برای بخش $x>L$:

$$\frac{d^2T_3}{dx^2}-\frac{\rho vc}{k}\frac{dT_3}{dx}=0,\ \theta_3=T_3-T_\infty \Longrightarrow \frac{d^2\theta_3}{dx^2}-\frac{\rho vc}{k}\frac{d\theta_3}{dx}=0$$

$$BC\begin{cases}(1)\ \ T_1(-\infty)=T_0 \Longrightarrow \theta_1(-\infty)=0\\ (2)\ \ T_1(0)=T_2(0) \Longrightarrow \theta_1(0)-\theta_2(0)=T_\infty-T_0\\ (3)\ \ T_2(L)=T_3(L) \Longrightarrow \theta_2(L)=\theta_3(L)\\ (4)\ \ \frac{dT_1(0)}{dx}=\frac{dT_2(0)}{dx} \Longrightarrow \frac{d\theta_1(0)}{dx}=\frac{d\theta_2(0)}{dx}\\ (5)\ \ \frac{dT_2(L)}{dx}=\frac{dT_3(L)}{dx} \Longrightarrow \frac{d\theta_2(L)}{dx}=\frac{d\theta_3(L)}{dx}\\ (6)\ \ T_3(\infty) \Longrightarrow finite \Longrightarrow \theta_3(\infty) \Longrightarrow finite\end{cases}$$

برای $x<0 \Longrightarrow \frac{d^2\theta_1}{dx^2}-m^2\frac{d\theta_1}{dx}=0,\ m^2=\frac{\rho vc}{k}$

فرض: $\theta_1=e^{\lambda x} \Longrightarrow \lambda^2-m^2\lambda=0 \Longrightarrow \lambda_1=0\ ,\ \lambda_2=m^2$

$$\theta_1(x)=Ae^{\lambda_1 x}+Be^{\lambda_2 x}=A+Be^{m^2x}$$

برای $0<x<L \quad \Longrightarrow \frac{d^2\theta_2}{dx^2}-m^2\frac{d\theta_2}{dx}-n^2\theta_2=0,\ n^2=\frac{hP}{kA},\ m^2=\frac{\rho vc}{k}$

فرض: $\theta_2=e^{\lambda x} \Longrightarrow \lambda^2-m^2\lambda-n^2=0 \Longrightarrow \lambda_{1,2}=\frac{m^2\pm\sqrt{m^4+4n^2}}{2}$

$$\theta_2(x)=Ce^{\left(\frac{m^2+\sqrt{m^4+4n^2}}{2}x\right)}+De^{\left(\frac{m^2-\sqrt{m^4+4n^2}}{2}x\right)}$$

برای $x>L \quad \Longrightarrow \frac{d^3\theta_3}{dx^2}-m^2\frac{d\theta_3}{dx}=0,\ m^2=\frac{\rho vc}{k}$

فرض: $\theta_3=e^{\lambda x} \Longrightarrow \lambda^2-m^2\lambda=0 \Longrightarrow \lambda_1=0\ ,\ \lambda_2=m^2$

$$\theta_3(x)=E\exp(0)+F\exp(m^2x)=E+Fe^{m^2x}$$

$(1) \Longrightarrow A+B\times 0=0 \Longrightarrow A=0$

$(2) \Longrightarrow B-(C+D)=T_\infty-T_0$

$(3) \Longrightarrow Ce^{\lambda_1 L}+De^{\lambda_2 L}-\left(E+Fe^{m^2L}\right)=T_\infty-T_0$

$(4) \Longrightarrow Bm^2=C\lambda_1+D\lambda_2$

$$(5) \Rightarrow C\lambda_1 e^{\lambda_1 L} + D\lambda_2 e^{\lambda_2 L} = Fm^2 e^{m^2 L}$$

$$(6) \Rightarrow F = 0$$

$$(5) \Rightarrow C\lambda_1 e^{\lambda_1 L} + D\lambda_2 e^{\lambda_2 L} = 0 \Rightarrow D = -c\frac{\lambda_1}{\lambda_2} e^{(\lambda_1-\lambda_2)L}$$

$$(4) \Rightarrow Bm^2 = C\lambda_1 + C\lambda_1 e^{(\lambda_1-\lambda_2)L} \Rightarrow B = \frac{C\lambda_1\left(1-e^{(\lambda_1-\lambda_2)L}\right)}{m^2}$$

$$(2) \Rightarrow \frac{C\lambda_1\left(1-e^{(\lambda_1-\lambda_2)L}\right)}{m^2} - C - C\frac{\lambda_1}{\lambda_2} e^{(\lambda_1-\lambda_2)L} = T_\infty - T_0$$

$$C\left(\frac{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}{m^2\lambda_2}\right) = T_\infty - T_0$$

$$\Rightarrow C = \frac{(T_\infty - T_0)m^2\lambda_2}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\Rightarrow D = -\frac{m^2\lambda_1(T_\infty - T_0)}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\Rightarrow B = \frac{\lambda_2(T_\infty - T_0)\lambda_1\left(1-e^{(\lambda_1-\lambda_2)L}\right)}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$(3) \Rightarrow Ce^{\lambda_1 L} + De^{\lambda_2 L} - E = T_\infty - T_0$$

$$\Rightarrow E = T_0 - T_\infty + \frac{m^2\lambda_2(T_\infty - T_0)e^{\lambda_1 L}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$- \frac{m^2\lambda_1(T_\infty - T_0)e^{\lambda_2 L}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\theta_1(x) = \frac{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right)e^{m^2x}(T_\infty - T_0)}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\Rightarrow \frac{T_1(x) - T_0}{T_\infty - T_0} = \frac{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right)e^{m^2x}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

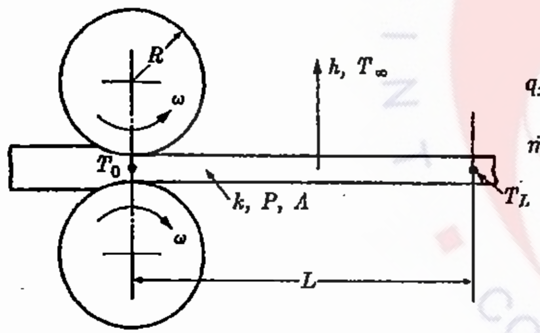
$$\theta_2(x) = T_2(x) - T_\infty = \frac{m^2\lambda_2(T_\infty - T_0)e^{\lambda_1 L}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

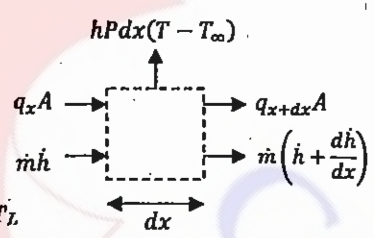
$$- \frac{m^2\lambda_1(T_\infty - T_0)e^{\lambda_2 L}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\Rightarrow \frac{T_2(x) - T_\infty}{T_0 - T_\infty} = \frac{m^2\lambda_1 e^{\lambda_2 x} - m^2\lambda_2 e^{\lambda_1 x}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

$$\theta_3(x) = E \Rightarrow \frac{T_3(x) - T_\infty}{T_0 - T_\infty} = 1 - \frac{m^2\lambda_2 e^{\lambda L}}{\lambda_1\lambda_2\left(1-e^{(\lambda_1-\lambda_2)L}\right) - m^2\lambda_2 + m^2\lambda_1 e^{(\lambda_1-\lambda_2)L}}$$

مساله ۲۶-۳)





$$q_xA - \left(q_x + \frac{dq_x}{dx}dx\right)A + \dot{m}\dot{h} - \dot{m}\left(\dot{h} + \frac{d\dot{h}}{dx}\right) - hPdx(T - T_\infty) = 0$$

$$-\frac{dq_x}{dx}dxA - \rho vA\frac{d\dot{h}}{dx}dx - hPdx(T - T_\infty) = 0$$

$$q_x = -k\frac{dT}{dx}, \dot{h} = CT \Rightarrow kA\frac{d^2T}{dx^2} - \rho vAc\frac{dT}{dx} - hP(T - T_\infty) = 0$$

$$\theta = T - T_\infty \Rightarrow \frac{d^2\theta}{dx^2} - \frac{\rho vc}{k}\frac{d\theta}{dx} - \frac{h\theta}{k\lambda} = 0\ , \frac{P}{A} = \frac{1}{\lambda}$$

$$BC\begin{cases} x = 0 \Rightarrow T = T_\infty \Rightarrow \theta(0) = T_0 - T_\infty \\ x \to \infty \Rightarrow T = finite \Rightarrow \theta(\infty) \Rightarrow finite \\ x = L \Rightarrow T = T_L \Rightarrow \theta(L) = T_L - T_\infty \end{cases}$$

فرض: $\theta = e^{ax}$

$$\Rightarrow a^2 - \frac{\rho vc}{k}a - \frac{h}{k\lambda} = 0 \Rightarrow a_{1,2} = \frac{1}{2}\frac{\rho vc}{k}\left(1 \pm \sqrt{1 + \frac{4hk^2}{\rho^2v^2c^2}}\right)$$

$$\Rightarrow \theta(x) = c_1e^{a_1x} + c_2e^{a_2x}$$

(2) $\Rightarrow \theta(\infty) =$ محدود $\Rightarrow c_1 = 0$

$$(1) \Rightarrow \theta(0) = T_0 - T_\infty \Rightarrow c_2 = T_0 - T_\infty \Rightarrow \frac{T(x) - T_\infty}{T_0 - T_\infty} = \exp\left[\frac{\rho vc}{k\lambda}\left(1 - \sqrt{1 + \frac{4hk^2}{\rho^2v^2c^2}}\right)x\right]$$

$$(3) \Rightarrow x = L \Rightarrow T = T_L \Rightarrow \frac{T_L - T_\infty}{T_0 - T_\infty} = \exp\left[\frac{\rho v c}{k\lambda} - \left(\sqrt{\left(\frac{\rho v c}{k\lambda}\right)^2 + \frac{h}{k}}\right) L\right]$$

$$\Rightarrow \frac{1}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty} = \frac{\rho v c}{k\lambda} - \sqrt{\left(\frac{\rho v c}{\lambda k}\right)^2 + \frac{h}{k}}, \beta = \frac{1}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}, \alpha = \frac{\rho v c}{\lambda k}$$

$$(\beta - \alpha)^2 = \alpha^2 + \frac{h}{\lambda} \Rightarrow \beta^2 + \alpha^2 - 2\beta\alpha = \alpha^2 + \frac{h}{\lambda} \Rightarrow 2\beta\alpha = \beta^2 - \frac{h}{\lambda}$$

$$\Rightarrow \alpha = \frac{\lambda\beta^2 - h}{2\beta} \Rightarrow \frac{\rho v c}{2k} = \frac{\frac{A}{P}\left(\frac{1}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}\right)^2 - h}{\frac{2}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}}$$

$$v = R\omega = \frac{\frac{2kA}{\rho c P}\left(\frac{1}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}\right)^2 - \frac{2hk}{\rho c}}{\frac{2}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}} \Rightarrow \omega = \frac{\frac{2kA}{\rho c P}\left(\frac{1}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}\right)^2 - \frac{hk}{\rho c}}{\frac{2R}{L} Ln \frac{T_L - T_\infty}{T_0 - T_\infty}}$$

مساله ۲۷-۳)

ابتدا معادله کلی حاکم بر پره‌ها را می‌نویسیم:

$$q_x A - q_x A - \frac{d}{dx}(q_x A)dx - hPdx(T - T_\infty) = 0$$

$$\Rightarrow -\frac{d}{dx}\left(-kA\frac{d_T}{dx}\right)dx - hPdx(T - T_\infty) = 0 \Rightarrow \frac{d}{dx}\left(A\frac{dT}{dx}\right) - \frac{hP}{k}(T - T_\infty) = 0$$

معادله حاکم بر تمام پره‌ها $\theta = T - T_\infty \Rightarrow \frac{d}{dx}\left(A\frac{dT}{dx}\right) - \frac{hP}{k}\theta = 0$

$$BC\begin{cases}(1)\ T(0) = \text{محدود} \Rightarrow \theta(0) = \text{محدود} \ \text{یا}\ \frac{d\theta(0)}{dx} = 0\\ (2)\ T(L) = T_0 \ \text{یا}\ \theta(L) = T_0 - T_\infty\end{cases}$$

a)

$y = Cx^{\frac{1}{2}} \Rightarrow A = 2Cx^{\frac{1}{2}}$, $P = 2\delta + 2w$ فرض: $\delta \ll$ و $w = 1$

$$\Rightarrow P = 2 \Rightarrow \frac{d}{dx}\left(2Cx^{\frac{1}{2}}\frac{d\theta}{dx}\right) - \frac{2h}{k}\theta = 0$$

$$\Rightarrow \frac{h}{ck} = m^2 \Rightarrow \frac{d}{dx}\left(x^{\frac{1}{2}}\frac{d\theta}{dx}\right) - m^2\theta = 0$$

تابع بسل استاندارد:

$$\frac{d}{dx}\left(x^\alpha \frac{dy}{dx}\right) + \gamma^2 x^\beta y = 0 \Rightarrow \beta = 0\ ,\ \gamma = \pm im\ ,\ \alpha = \frac{1}{2}$$

$$\beta - \alpha + 2 = 0 - \frac{1}{2} + 2 = \frac{3}{2} \qquad v = \frac{1-\alpha}{\beta - \alpha + 2} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \quad \mu = \frac{2}{\beta - \alpha + 2} = \frac{4}{3}$$

$$\Rightarrow \theta(x) = c_1 x^{\frac{1}{3}} I_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right) + c_2 x^{\frac{1}{4}} k_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right)$$

$$\theta(0) = \text{محدود} \Rightarrow c_2 = 0$$

$$\theta(L) = T_0 - T_\infty \Rightarrow T_0 - T_\infty = c_1 L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right) \Rightarrow c_1 = \frac{T_0 - T_\infty}{L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right)}$$

$$\Rightarrow \theta(x) = T(x) - T_\infty = (T_0 - T_\infty)\left(\frac{x}{L}\right)^{\frac{1}{4}} \frac{I_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4m}{3}L^{\frac{3}{4}}\right)}$$

b)

$$y = Cx^2 \Rightarrow A = 2Cx^2\ , P = 2 \Rightarrow \frac{d}{dx}\left(A\frac{d\theta}{dx}\right) - \frac{hP}{k}\theta = 0 \Rightarrow$$

$$\frac{d}{dx}\left(2Cx^2\frac{d\theta}{dx}\right) - \frac{2h}{k}\theta = 0 \Rightarrow \frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) - \frac{h}{k}\theta = 0 \qquad , \qquad m^2 = \frac{h}{kc} \Rightarrow$$

$$\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) - m^2\theta = 0$$

تابع بسل استاندارد:

$$\frac{d}{dx}\left(x^\alpha \frac{dy}{dx}\right) + \gamma^2 x^\beta y = 0 \Rightarrow \beta = 0\ , \alpha = 2 \Rightarrow \beta - \alpha + 2 = 0$$

$$, \gamma = \pm im \Rightarrow \theta(x) = x^r \Rightarrow r^2 + (\alpha - 1)r + \gamma^2 = 0 \Rightarrow r^2 + (\alpha - 1)r - m^2 = 0$$

$$\Rightarrow r^2 + r - m^2 = 0 \Rightarrow r_{1,2} = \frac{1}{2}\left(-1 \pm \sqrt{1 + 4m^2}\right) \Rightarrow \begin{matrix} r_1 > 0 \\ r_2 < 0 \end{matrix} \Rightarrow$$

$$\theta(x) = c_1 x^{-\frac{1}{2} + \sqrt{\frac{1+4m^2}{2}}} + c_2 x^{-\frac{1}{2} - \sqrt{\frac{1+4m^2}{2}}}$$

$$\theta(x) = \text{محدود} \Rightarrow c_2 = 0\ , \theta(L) = T_0 - T_\infty$$

$$\Rightarrow c_1 L^{\left(-\frac{1}{2} + \frac{\sqrt{1+4m^2}}{2}\right)} = T_0 - T_\infty \Rightarrow c_1 (T_0 - T_\infty) L$$

$$\Rightarrow A(x) = (T_0 - T_\infty)x^{\frac{-1+\frac{\sqrt{1+4m^2}}{2}}{2}} . L^{\frac{1-\frac{\sqrt{1+4m^2}}{2}}{2}} \Rightarrow \frac{T(x)-T_\infty}{T_0-T_\infty} = \left(\frac{x}{L}\right)^{\frac{-1+\frac{\sqrt{1+4m^2}}{2}}{2}}$$

$$\Rightarrow \frac{T(x)-T_\infty}{T_0-T_\infty} = \left(\frac{x}{L}\right)^{\frac{-1+\sqrt{1+\frac{4h}{kc}}}{2}}$$

در حالت a:

$$\frac{T(x)-T_\infty}{T_0-T_\infty} = \left(\frac{x}{L}\right)^{\frac{1}{4}} \frac{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)} \Rightarrow \left.\frac{dT}{dx}\right|_{x=L} = \frac{1}{4L}\left(\frac{x}{L}\right)^{-\frac{3}{4}} \frac{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}$$

$$-mx^{-\frac{1}{4}}\left(\frac{x}{L}\right)^{\frac{1}{4}} \frac{I_{-\frac{2}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)} - \frac{1}{4mx^{\frac{3}{4}}}\left(\frac{x}{L}\right)^{\frac{1}{4}} \frac{I_{-\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3}mx^{\frac{3}{4}}\right)}$$

$$k_{\frac{1}{3}}(u) \sim \left(\frac{\pi}{2U}\right)^{\frac{1}{2}} e^{-U}\left(1 + \frac{4\left(\frac{1}{3}\right)^2 - 1^2}{1!\,8U} + \frac{\left(4\left(\frac{1}{3}\right)^2-1\right)\left(4\left(\frac{1}{3}\right)^2-3^2\right)}{2!\,(8U)^2} + \cdots\right)$$

$$x^{\frac{1}{4}}k_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right) \sim x^{\frac{1}{4}}\left(\frac{3\pi}{8mx^{\frac{3}{4}}}\right)^{\frac{1}{4}} e^{\frac{-4m}{3}x^{\frac{3}{4}}}\left(1 + \frac{\left(4\left(\frac{1}{3}\right)^2-1\right)}{0!\left(\frac{32mx^{\frac{3}{4}}}{3}\right)} + \frac{\left(4\left(\frac{1}{3}\right)^2-1\right)\left(4\left(\frac{1}{3}\right)^2-3^2\right)}{2!\left(\frac{32mx^{\frac{3}{4}}}{3}\right)^2} + \cdots\right)$$

$$x \to 0 \Rightarrow x^{\frac{1}{4}}k_{\frac{1}{3}}\left(\frac{4m}{3}x^{\frac{3}{4}}\right) \to \infty \Rightarrow \theta(0) = \text{محدود} \Rightarrow c_2 = 0$$

$$\Rightarrow \left.\frac{dT}{dx}\right|_{x=L} = (T_0 - T_\infty)\left(\frac{1}{4L} - \frac{m}{L^{\frac{1}{4}}}\frac{I_{-\frac{2}{3}}\left(\frac{4}{3}ml^{\frac{3}{4}}\right)}{I_{-\frac{2}{3}}\left(\frac{4}{3}ml^{\frac{3}{4}}\right)} - \frac{1}{4ml^{\frac{3}{4}}}\right)$$

$$Q = -kA\left.\frac{dT}{dx}\right|_{x=L} = k(2b)\left(\frac{1}{4L} - \frac{m}{L^{\frac{1}{4}}}\frac{I_{-\frac{2}{3}}\left(\frac{4}{3}ml^{\frac{3}{4}}\right)}{I_{-\frac{1}{3}}\left(\frac{4}{3}ml^{\frac{3}{4}}\right)} - \frac{1}{4ml^{\frac{3}{4}}}\right)(T_\infty - T_0)$$

در حالت b:

$$T(x) - T_\infty = (T_0 - T_\infty)\left(\frac{x}{L}\right)^{\frac{-1+\sqrt{1+\frac{4h}{kc}}}{2}} \Rightarrow \left.\frac{dT}{d_x}\right|_{x=L} = (T_0 - T_\infty)\left(\frac{\sqrt{\frac{4h}{kc}+1}-1}{2L}\right)$$

$$\Rightarrow Q = -kA\left.\frac{dT}{dx}\right|_{x=L} \Rightarrow Q = \frac{bk}{L}(T_\infty - T_0)\left(\sqrt{1+\frac{4h}{kc}} - 1\right)$$

مسأله ۲۸-۳)

(a

$$yr^{\frac{1}{2}} = C \quad \text{at } r = R_i \to y = b \to bR_i^{\frac{1}{2}} = C \Rightarrow yr^{\frac{1}{2}} = bR_i^{\frac{1}{2}}$$

$$(q_r + q_{r+dr})A_r - 2A_s h(T - T_\infty) = 0\,,\ A_r = 2\pi r.2y$$

$$\Rightarrow -\frac{d}{dr}(2\pi r.2yq_r) - 4\pi rh(T - T_\infty) = 0\,,\ A_s = 2\pi r.dr$$

$$\Rightarrow \frac{d}{dr}\left(bR_i^{\frac{1}{2}}r^{\frac{1}{2}}k\frac{dT}{dr}\right) - rh(T - T_\infty) = 0 \Rightarrow \frac{d}{dr}\left(r^{\frac{1}{2}}\frac{d\theta}{dr}\right) - \frac{h}{bR_i^{0.5}k}r\theta = 0$$

$$m^2 = \frac{h}{bR_i^{0.5}k}$$

$$\alpha = \frac{1}{2}\,,\ \beta = 1\,,\ \gamma^2 = -m^2 \to \beta - \alpha + 2 = \frac{5}{2}\,,\ v = \frac{1}{5}\,,\ \mu = \frac{4}{5}\,,\ \frac{v}{\mu} = \frac{1}{4}$$

$$\Rightarrow \theta(r) = r^{\frac{1}{4}}\left[c_1 I_{\frac{1}{5}}\left(\frac{4}{5}mr^{\frac{5}{4}}\right) + c_2 I_{-\frac{1}{5}}\left(\frac{4}{5}mr^{\frac{5}{4}}\right)\right],\ \text{BC}\begin{cases}\theta(R_i) = \theta_0\\ \frac{d\theta(R_0)}{dr} = 0\end{cases}$$

$$\frac{d\theta(R_0)}{dr} = 0 \Rightarrow$$

$$\frac{1}{4}R_0^{-\frac{5}{4}}\left[c_1 I_{\frac{1}{5}}\left(\frac{4}{5}mR_0^{\frac{5}{4}}\right) + c_2 I_{-\frac{1}{5}}\left(\frac{4}{5}mR_0^{\frac{5}{4}}\right)\right] + R_0^{\frac{1}{4}}\left[c_1 mR_0^{\frac{5}{4}}.I_{-\frac{4}{5}}\left(\frac{4}{3}mR_0^{\frac{5}{4}}\right) +\right.$$

$$\left.c_2 mR_0^{\frac{5}{4}}.I_{\frac{4}{5}}\left(\frac{4}{3}mR_0^{\frac{5}{4}}\right)\right] = 0 \ (1)$$

$$\theta_0 = \theta(R_i) \Rightarrow R_i^{\frac{1}{4}}\left[c_1 I_{\frac{1}{5}}\left(\frac{4}{3}mR_i^{\frac{5}{4}}\right) + c_2 I_{-\frac{1}{5}}\left(\frac{4}{3}mR_i^{\frac{5}{4}}\right)\right] = \theta_0 \ (2)$$

$$\begin{cases} c_1 \underbrace{\left[\frac{1}{4}R_0^{-\frac{5}{4}} I_{\frac{1}{5}}\left(\frac{4}{5}mR_0^{\frac{5}{4}}\right) + R_0^{\frac{1}{4}} mR_0^{\frac{5}{4}}.I_{-\frac{4}{5}}\left(\frac{4}{3}mR_0^{\frac{5}{4}}\right)\right]}_{\beta_1} \\ \qquad + c_2 \underbrace{\left[\frac{1}{4}R_0^{-\frac{5}{4}} I_{-\frac{1}{5}}\left(\frac{4}{5}mR_0^{\frac{5}{4}}\right) + R_0^{\frac{1}{4}} mR_0^{\frac{5}{4}}.I_{\frac{4}{5}}\left(\frac{4}{3}mR_0^{\frac{5}{4}}\right)\right]}_{\beta_2} = 0 \\ c_1 \underbrace{\left[R_i^{\frac{1}{4}} I_{\frac{1}{5}}\left(\frac{4}{3}mR_i^{\frac{5}{4}}\right)\right]}_{\beta_3} + c_2 \underbrace{\left[R_i^{\frac{1}{4}} I_{-\frac{1}{5}}\left(\frac{4}{3}mR_i^{\frac{5}{4}}\right)\right]}_{\beta_4} = \theta_0 \end{cases}$$

$$\Longrightarrow c_1 = \frac{1}{\beta_1}\left[\theta_0 + \frac{\beta_2\beta_3\theta_0}{\beta_1\beta_4-\beta_2\beta_3}\right],\ c_2 = \frac{-\beta_3\theta_0}{\beta_1\beta_4-\beta_2\beta_3}$$

$$q_t = -kA\frac{d\theta}{dr}\bigg|_{r=R_i} = -k(2\pi R_i.2b)\frac{d\theta}{dr}\bigg|_{r=R_l}$$

b)

$$yr^2 = C \text{ در } r = R_i \to y = b\,,\ bR_i^2 = C \to yr^2 = bR_i^2$$

$$-\frac{d}{dr}(ryq_r) - rh(T-T_\infty) = 0 \to \frac{d}{dr}\left(\frac{bR_i^2}{r}k\frac{dT}{dr}\right) - rh(T-T_\infty) = 0$$

$$\Longrightarrow \frac{d}{dr}\left(\frac{1}{r}\frac{d\theta}{dr}\right) - \frac{hr}{bR_i^2k}\theta = 0\,,\ m^2 = \frac{h}{bR_i^2k}$$

$$\alpha = -1\,,\ \beta = 1\ ,\ \gamma^2 = -m^2\,,\ \beta-\alpha+2 = 4\,,\ v = \frac{1}{2}\,,\ \mu = \frac{1}{2}$$

$$\Longrightarrow \theta(r) = r\left[c_1 I_{\frac{1}{2}}\left(\frac{m}{2}r^2\right) + c_2 I_{\frac{1}{2}}\left(\frac{m}{2}r^2\right)\right]$$

$$\frac{d\theta(R_0)}{dr} =$$

$$\left[c_1 I_{\frac{1}{2}}\left(\frac{mR_0^2}{2}\right) + c_2 I_{-\frac{1}{2}}\left(\frac{mR_0^2}{2}\right)\right] + R_0\left[c_1 m I_{-\frac{1}{2}}\left(\frac{mR_0^2}{2}\right) + c_2 m I_{\frac{1}{2}}\left(\frac{mR_0^2}{2}\right)\right] = 0 \quad (1)$$

$$\theta_0 = R_i\left[c_1 I_{\frac{1}{2}}\left(\frac{mR_i^2}{2}\right) + c_2.I_{-\frac{1}{2}}\left(\frac{mR_i^2}{2}\right)\right] \quad (2)$$

$$\begin{cases} c_1 \underbrace{\left[I_{\frac{1}{2}}\left(\frac{mR_0^2}{2}\right) + R_0 m I_{-\frac{1}{2}}\left(\frac{mR_0^2}{2}\right)\right]}_{\beta_1} + c_2 \underbrace{\left[I_{-\frac{1}{2}}\left(\frac{mR_0^2}{2}\right) + R_0 m I_{\frac{1}{2}}\left(\frac{mR_0^2}{2}\right)\right]}_{\beta_2} = 0 \\ c_1 \underbrace{\left[R_i I_{\frac{1}{2}}\left(\frac{mR_i^2}{2}\right)\right]}_{\beta_3} + c_2 \underbrace{\left[R_i.I_{-\frac{1}{2}}\left(\frac{mR_i^2}{2}\right)\right]}_{\beta_4} = \theta_0 \end{cases}$$

$$q_t = -kA\frac{d\theta}{dr}\bigg|_{r=R_l} = -k(2\pi R_i.2b)\frac{d\theta}{dr}\bigg|_{r=R_i}$$

مساله ۲۹-۳)

a)

$$y = Cx \qquad b = cL \Longrightarrow c = \frac{b}{L} \Longrightarrow y = \frac{b}{L}x$$

$$-\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) - hS(T-T_\infty) = 0$$

$$A = \pi y^2\,,\ S = 2\pi y ds\,,\ \cos\theta = \frac{dx}{ds} = \frac{L}{\sqrt{L^2+b^2}} \Longrightarrow ds = \frac{\sqrt{L^2+b^2}}{L}dx$$

$$-\frac{d}{dx}\left(-k\pi\left(\frac{b}{L}\right)^2 x^2\frac{dT}{dx}\right)dx - h2\pi\frac{b}{L}x\frac{\sqrt{L^2+b^2}}{L}dx(T-T_\infty) = 0$$

$$\Longrightarrow \frac{d}{dx}\left(x^2\frac{dT}{dx}\right) - \frac{2h\sqrt{L^2+b^2}}{kb}x(T-T_\infty) = 0$$

$$m^2 = \frac{2h\sqrt{L^2+b^2}}{kb} \quad , \quad \theta = T - T_\infty$$

$$\Longrightarrow \frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) - m^2\theta = 0$$

$$\alpha = 2\ ,\ \beta = 1 \Longrightarrow \beta-\alpha+2 = 1 \neq 0 \to \text{تابع بسل}$$

$$v = \frac{1-\alpha}{\beta-\alpha+2} = -1\,,\ \mu = \frac{2}{\beta-\alpha+2}\,,\ \frac{v}{\mu} = -\frac{1}{2}\,,\ \gamma^2 = -m^2$$

$$\Longrightarrow \theta(x) = T(x) - T_\infty =$$

$$Ax^{-\frac{1}{2}}I_{-1}\left(2\sqrt{\frac{2h\sqrt{L^2+b^2}}{kb}}\,x^{\frac{1}{2}}\right) + Bx^{-\frac{1}{2}}k_{-1}\left(2\sqrt{\frac{2h\sqrt{L^2+b^2}}{kb}}\,x^{\frac{1}{2}}\right)$$

$$BC\begin{cases} x = 0\ :\ \theta = finite \Longrightarrow B = 0 \\ x = L\ :\ \theta = \theta_0 \Longrightarrow \dfrac{\theta(x)}{\theta_0} = \left(\dfrac{x}{L}\right)^{-\frac{1}{2}}\dfrac{I_{-1}\left(2\sqrt{\frac{2h\sqrt{L^2+b^2}}{kb}}\,x^{\frac{1}{2}}\right)}{I_{-1}\left(2\sqrt{\frac{2h\sqrt{L^2+b^2}}{kb}}\,L^{\frac{1}{2}}\right)} \end{cases}$$

b)

$$y = be^{-nx}$$

با استفاده از مساله (الف) و صرفنظر از انحنا برای به دست آوردن P خواهیم داشت:

$$-\frac{d}{dx}\left(-k\pi(be^{-nx})^2\frac{dT}{dx}\right)dx - h.2\pi be^{-nx}dx(T-T_\infty)=0$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2nx}\frac{dT}{dx}\right)-\frac{2h}{bk}e^{nx}(T-T_\infty)=0$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2nx}\frac{d\theta}{dx}\right)-\frac{2h}{bk}e^{-nx}\theta=0 \quad BC\begin{cases}x=0\ :\ \theta=finite\\ x=L\ :\ \theta=\theta_0\end{cases}$$

$$e^{-nx}=u\ \ dx=-ne^{-nx}dx \Rightarrow -nudx=du\rightarrow dx=\frac{du}{-nu}$$

$$-nu\frac{d}{du}\left(-\frac{1}{n}u\frac{d\theta}{du}\right)-\frac{2h}{kb}u\theta=0\rightarrow\frac{d}{du}\left(u\frac{d\theta}{du}\right)-\frac{2h}{kb}\theta=0$$

$\alpha=1,\beta=0\ ,\ x^2=\frac{2h}{kb}=\gamma^2\rightarrow\beta-\alpha+2=1\neq 0\rightarrow$ تابع بسل

$$v=\frac{1-\alpha}{\beta-\alpha+2}\ \ ,\ \ \mu=2\ ,\ \frac{v}{\mu}=0$$

$$\theta(u)=AI_0\left(2\sqrt{\frac{2h}{kb}}u^{\frac{1}{2}}\right)+Bk_0\left(2\sqrt{\frac{2h}{kb}}u^{\frac{1}{2}}\right)$$

$$\theta(e^{-nx})=AI_0\left(2\sqrt{\frac{2h}{kb}}e^{-\frac{nx}{2}}\right)+Bk_0\left(2\sqrt{\frac{2h}{kb}}e^{-\frac{nx}{2}}\right)$$

$$x=0\rightarrow\theta=finite\rightarrow B=0$$

$$x=L\rightarrow\theta=\theta_0\rightarrow\frac{\theta}{\theta_0}=\frac{I_0\left(2\sqrt{\frac{2h}{kb}}e^{\frac{-nx}{2}}\right)}{I_0\left(2\sqrt{\frac{2h}{kb}}e^{\frac{-nx}{2}}\right)}$$

مساله ۳۰-۳)

برای دیسک بالایی:

$$q_r(2\pi r)\delta_1|_r-q_r(2\pi r)\delta_1|_{r+dr}-h_1 2\pi rdr(T_1-T_\infty)+\mu PV(2\pi rdr)=0$$

$$\frac{d}{dr}(rq_r)-\frac{h_1r}{\delta_1}(T_1-T_\infty)+\frac{\mu PV}{\delta_1}r=0$$

$$\Rightarrow\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)-\frac{h_1r}{k_1\delta_1}(T_1-T_\infty)+\frac{\mu PV}{k_1\delta_1}r=0$$

$$\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)-\frac{h_1}{k_1\delta_1}(T_1-T_\infty)+\frac{\mu PV}{k_1\delta_1}=0\ ,V=R\omega$$

$$BC\begin{cases}r=0\ :\ T_1=finite\Rightarrow\frac{dT_1}{dr}=0\\ r=R\ :\ -k_1(2\pi R)\delta_1\Rightarrow\frac{dT_1}{d_r}=h_3(2\pi R)\delta_1(T_1-T_\infty)\\ \qquad\Rightarrow -k_1\frac{dT_1}{dr}=h_3(T_1-T_\infty)\end{cases}$$

$$\theta_1=T_1-T_\infty\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta_1}{dr}\right)-\frac{h_1\theta_1}{k_1\delta_1}+\frac{\mu P(R\omega)}{k_1\delta_1}=0$$

$$\Rightarrow BC\begin{cases}r=0\ :\ \theta_1=finite\Rightarrow\frac{d\theta_1}{dr}=0\\ r=R\ :\ -k_1\frac{d\theta_1}{dr}=h_3\theta_1\end{cases}$$

$$r^2\frac{d^2\theta_1}{dr^2}+\frac{d\theta_1}{dr}-\frac{h_1}{k_1\delta_1}r^2\theta_1=\frac{-\mu P(R\omega)}{k_1\delta_1}r^2$$

$$\theta_{1,h}=c_1I_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)+c_2k_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)$$

$$\theta_{1,p}=a_0+a_1r+a_2r^2\Rightarrow\theta_{1,p}=\frac{\mu P(R\omega)}{h_1}$$

$$\theta_1(r)=\theta_{1,h}+\theta_{1,p}=c_1I_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)+c_2k_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)+\frac{\mu P(R\omega)}{h_1}$$

$$BC\,1:\ r=0\Rightarrow\theta_1=finite\Rightarrow c_2=0$$

$$BC\,2:\ r=R\Rightarrow -k\frac{d\theta_1}{dr}\Big|_R=h_3\,\theta_1|_R$$

$$-k_1c_1\frac{d}{dr}\left(I_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)\right)\Bigg|_{r=R}=h_3\left[c_1I_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}R\right)+\frac{\mu P(R\omega)}{h_1}\right]\Rightarrow c_1$$

$$\Rightarrow\theta_1(r)=T_1(r)-T_\infty=c_1I_0\left(\sqrt{\frac{h_1}{k_1\delta_1}}r\right)+\frac{\mu P(R\omega)}{h_1}$$

برای دیسک پایینی:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)-\frac{h_2}{k_2\delta_2}(T_2-T_\infty)+\frac{\mu PV}{k_2\delta_2}=0$$

$$\theta_2=T_2-T_\infty\qquad BC\begin{cases}r=0\ :\ T_2=finite\Rightarrow\theta_2=finite\\ r=R\ :\ -k_2k\frac{d\theta_2}{dr}\Big|_R=h_4\theta_2|_R\end{cases}$$

$$\Longrightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta_2}{dr}\right)-\frac{h_2\theta_2}{k_2\delta_2}=\frac{-\mu P(R\omega)}{k_2\delta_2}\Longrightarrow r^2\frac{d^2\theta_2}{dr^2}+\frac{d\theta_2}{dr}-\frac{h_2}{k_2\delta_2}r^2\theta=\frac{-\mu P(R\omega)}{k_2\delta_2}r^2$$

$$\Longrightarrow \theta_2(r)=c_3I_0\left(\sqrt{\frac{h_2}{k_2\delta_2}}r\right)+c_4k_0\left(\sqrt{\frac{h_2}{k_2\delta_2}}r\right)+\frac{\mu P(R\omega)}{h_2}$$

$$BC\begin{cases} r=0\ :\ \theta_2=finite\Longrightarrow c_4=0\\ r=R\ :\ -c_3\frac{d}{dr}\left(I_0\left(\sqrt{\frac{h_2}{k_2\delta_2}}r\right)\right)\Big|_{r=R}=h_4\left[c_3I_0\left(\sqrt{\frac{h_2}{k_2\delta_2}}R\right)+\frac{\mu P(R\omega)}{h_2}\right]\end{cases}$$

$\Longrightarrow$ c_3 به دست خواهد آمد

مساله ۳۲-۳)

برای دیواره کتری:

$$(q_y-q_{y+dy})-h_1\left(2\pi\left(R+\frac{\delta}{2}\right)dy\right)(T_1-T_\infty)-h_2\left(2\pi\left(R+\frac{\delta}{2}\right)dy\right)(T_1-T_\infty)=0$$

$$dx=dy\ ,\ q_x=q_y\cdot cos\alpha\ ,\ q_{x+dx}=q_{y+dy}\cdot cos\alpha$$

$$\Longrightarrow \frac{1}{cos\alpha}(q_x-q_{x+dx})-(h_1+h_2)2\pi\left(R+\frac{\delta}{2}\right)dx(T_1-T_\infty)=0$$

$$\frac{d}{dx}\left(k.A_x\frac{dT_1}{dx}\right)dx-(h_1+h_2)2\pi\left(R+\frac{\delta}{2}\right)cos\alpha\cdot dx(T_1-T_\infty)=0$$

$$A_x=2\pi\left(R+\frac{\delta}{2}\right)\delta$$

$$\theta_1=T_1-T_\infty\ ,\ m_1^2=\frac{h_1+h_2}{k\delta}cos\alpha\Longrightarrow\frac{d^2\theta_1}{dx^2}-m_1^2\theta_1=0\ (1)$$

$$\Longrightarrow\theta_1(x)=c_1\sinh(m_1x)+c_2\cosh(m_1x)$$

برای زیر کتری:

$$(q_r-q_{r+dr})-h_3(2\pi rdr)(T_2-T_\infty)+q''(2\pi rdr)=0$$

$$\frac{d}{dr}\left(k.A_r\frac{dT_2}{dr}\right)dr-h_32\pi rdr(T_2-T_\infty)+q''2\pi rdr=0$$

$$A_r=2\pi r\delta$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)-\frac{h_3}{k\delta}(T_2-T_\infty)+\frac{q''}{k\delta}=0$$

$$m_2^2=\frac{h_3}{k\delta},\theta_2=T_2-T_\infty\Longrightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta_2}{dr}\right)-m_2^2\theta_2+\underbrace{\left(\frac{q''}{k\delta}\right)}_{n}=0\ (2)$$

$$\Longrightarrow\theta_2(r)=c_3I_0(m_2r)+c_4k_0(m_2r)+\frac{n}{m_2^2}$$

$$BC\begin{cases}\frac{d\theta_1}{dx}(0)=0\\ \theta_1(Lcos\alpha)=\theta_2\left(R+\frac{\delta}{2}\right)\\ -k\frac{d\theta_1}{dx}(Lcos\alpha)=-k\frac{d\theta_2}{dr}\left(R+\frac{\delta}{2}\right)\\ \theta_2(0)=\text{finite}\quad or\quad\frac{d\theta_2}{dr}(0)=0\end{cases}$$

$$\frac{d\theta_1}{dx}(0)=0\Longrightarrow c_1=0\Longrightarrow\theta_1(x)=c_2\cosh(m_1x)$$

$$\frac{d\theta_2}{dr}(0)=0\Longrightarrow c_4=0\Longrightarrow\theta_2(r)=c_3I_0(m_2r)+\frac{q''}{h_3}$$

$$\theta_1(Lcos\alpha)=\theta_2\left(R+\frac{\delta}{2}\right)\Longrightarrow c_2\cosh(m_1Lcos\alpha)=c_3I_0\left(m_2\left(R+\frac{\delta}{2}\right)\right)+\frac{q''}{h_3}$$

$$-k\frac{d\theta_1}{dx}(Lcos\alpha)=-k\frac{d\theta_2}{dr}\left(R+\frac{\delta}{2}\right)$$

$$\Longrightarrow c_2m_1\sinh(m_1Lcos\alpha)=c_3\,m_2I_1\left(m_2\left(R+\frac{\delta}{2}\right)\right)$$

$$\Longrightarrow c_3=\frac{\frac{q''}{h_3}}{\frac{m_2}{m_1}I_1\left(m_2\left(R+\frac{\delta}{2}\right)\right)\coth(m_1Lcos\alpha)-I_0\left(m_2\left(R+\frac{\delta}{2}\right)\right)}$$

$$,\ c_2=\frac{m_2I_1\left(m_2\left(R+\frac{\delta}{2}\right)\right)}{m_1\sinh(m_1Lcos\alpha)}\cdot\frac{\frac{q''}{h_3}}{\frac{m_2}{m_1}I_1\left(m_2\left(R+\frac{\delta}{2}\right)\right)\coth(m_1Lcos\alpha)-I_0\left(m_2\left(R+\frac{\delta}{2}\right)\right)}$$

مساله ۳۳-۳)

$$q_r.A-q_r.A-\frac{d(q_rA)}{dr}dr-\dot{m}\dot{h}-\dot{m}\dot{h}-\frac{d}{dr}(\dot{m}\dot{h})dr-(h_1+h_2)2\pi dr(T-T_\infty)=0$$

$$\Longrightarrow-\frac{d}{dr}(q_rA)-\frac{d}{dr}(\dot{m}\dot{h})-(h_1+h_2)(2\pi r)(T-T_\infty)=0$$

$$q_r = -k\frac{dT}{dr}, A = 2\pi r\delta, \dot{h} = cT, \dot{m} = \rho v$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right) - \frac{\rho vc}{2k\pi\delta}\frac{dT}{dr} - \frac{(h_1+h_2)}{k\delta}r(T-T_\infty) = 0, T - T_\infty = \theta$$

$$\Rightarrow \frac{d^2\theta}{dr^2} + \left(1 - \frac{\rho vc}{2\pi k\delta}\right)\frac{d\theta}{dr} - \left(\frac{h_1+h_2}{k\delta}\right)r\theta = 0 \Rightarrow r\frac{d^2\theta}{dr^2} + a\frac{d\theta}{dr} - b^2 r\theta = 0$$

$$\Rightarrow r^2\frac{d^2\theta}{dr^2} + ar\frac{d\theta}{dr} - b^2r^2\theta = 0, a = \left(1 - \frac{\rho vc}{2k\pi\delta}\right), b^2 = \frac{h_1+h_2}{k\delta}$$

$$BC\begin{cases} T(R_i) = T_i \Rightarrow \theta(R_i) = T_i - T_\infty \\ T(\infty) = T_\infty \Rightarrow \theta(\infty) = 0 \end{cases} \quad v = \frac{1-q}{2} = \frac{\rho vc}{4\pi k\delta}, b = \sqrt{\frac{h_1+h_2}{k\delta}}$$

$$\theta = r^v y$$

$$\Rightarrow r^v\frac{d^2y}{dr^2} + (a+2v)r^{v-1}\frac{dy}{dr} + (-b^2r^v + [(a-1)v + v^2]r^{v-2})y = 0$$

$$\Rightarrow \theta(r) = c_1 r^v I_v(br) + c_2 r^v k_v(br)$$

$$a + 2v = 1 \Rightarrow r\frac{d}{dr}\left(r\frac{dy}{dr}\right) - (b^2r^2 + v^2)y = 0$$

$$y(r) = a_n I_0(br) + a_1 k_v(br)$$

$$(1) \Rightarrow \theta(\infty) = 0 \Rightarrow c_1 = 0$$

$$(2) \Rightarrow \theta(R_i) = T_i - T_\infty = c_2 R_i^v k_v(bR_i) \Rightarrow$$

$$c_1 = \frac{T_i - T_\infty}{R_i^v k_v(bR_i)} \Rightarrow \mathrm{T}(r) - T_\infty = \frac{T_i - T_\infty}{R_i^v k(vR_i)}k_v(br)$$

$$\frac{T(r) - T_\infty}{T_i - T_\infty} = \left(\frac{r}{R_i}\right)^v \frac{k_v(br)}{k_v(bR_i)} = \left(\frac{r}{R_i}\right)^{\frac{\rho vc}{4\pi k\delta}} \frac{k_v\left(\sqrt{\frac{h_1+h_2}{k\delta}}r\right)}{k_v(\sqrt{4}\,R_i)}$$

مساله ۳۵-۳)

فرمولاسیون متمرکز:

$$\mu PR\omega - 2\pi Rh(T - T_\infty) = \rho c(\pi R^2\delta)\frac{dT}{dt}, \theta = T - T_\infty$$

$$\Rightarrow \frac{d\theta}{dt} + \frac{2\pi Rh}{\rho c\pi R^2\delta}\theta = \frac{\mu PR\omega}{\rho c\pi R^2\delta} \Rightarrow \frac{d\theta}{dt} + \frac{2h}{\rho cR\delta}\theta = \frac{\mu P\omega}{\rho c\pi R\delta}$$

$$\Rightarrow \theta(T) = T(T) - T_\infty = \frac{\mu P\omega}{\rho c\pi R\delta}\left(1 - e^{-\frac{\mu P\omega t}{\rho c\pi R\delta}}\right)$$

فرمولاسیون دیفرانسیلی:

$$q_r.A|_r - q_r.A|_{r+dr} = \rho vc\frac{dT}{dt} \Rightarrow -\frac{d(q_r.A)}{dr}dr = \rho vc\frac{dT}{dt}$$

$$\Rightarrow \rho vc\frac{dT}{dt} = k\frac{d}{dr}\left(2\pi r\delta\frac{dT}{dr}\right)dr \Rightarrow \rho c.2\pi r.\delta dr.\frac{dT}{dr} = k\delta\frac{d}{dr}\left(2\pi r\frac{dT}{dr}\right)dr$$

$$,\theta = T - T_\infty, \alpha = \frac{k}{\rho c} \Rightarrow r\frac{dT}{dt} = \alpha\frac{d}{dr}\left(r\frac{dT}{dr}\right) \Rightarrow r\frac{d\theta}{dt} = \alpha\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)$$

$$BC\begin{cases} \frac{q''}{k} = \frac{d\theta(R,t)}{dr}\ (1) \\ T(0,t) = finite \Rightarrow \theta(0,t) = finite\ or\ \frac{d\theta(0,t)}{dr} = 0\ (2) \end{cases}$$

$$T(r,0) = T_\infty \Rightarrow \theta(r,0) = 0\ (3)$$

$$\theta(r,t) = \Psi(r,t) + \phi(r) + \Gamma(t) \Rightarrow \frac{d\Psi}{dt} = \frac{\alpha}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right)$$

$$\frac{d\Gamma}{dt} = \frac{\alpha}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$$

$$(3) \Rightarrow \theta(r,0) = 0 \Rightarrow 0 = \Psi(r,0) + \phi(r) + \Gamma(0)$$

$$\Rightarrow \Psi(r,0) = -\phi(r) - \Gamma(0)$$

$$(2) \Rightarrow \frac{d\Psi(0,t)}{dr} = 0 \quad , \quad \frac{d\phi(0)}{dr} = 0$$

$$(1) \Rightarrow k\frac{d\theta(R,t)}{dr} = q'' \Rightarrow \frac{d\Psi(R,t)}{dr} = 0, k\frac{d\phi(R)}{dr} = q''$$

$$\frac{1}{\alpha}\frac{d\Gamma}{dt} = c \Rightarrow \Gamma = \alpha ct + c_1$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = c \Rightarrow r\frac{d\phi}{dr} = \frac{cr^2}{2} + c_2 \Rightarrow \phi(r) = \frac{cr^2}{4} + c_2\ln r + c_3$$

$$\frac{d\phi(0)}{dr} = 0 \Rightarrow 0 = 0 + c_2 \Rightarrow c_2 = 0, k\frac{d\phi(R)}{dr} = q'' \Rightarrow C = \frac{2q''}{kR}$$

به صورت اختیاری: $c_1 + c_3 = 0 \Rightarrow \Gamma = \frac{2q''}{kR}t \quad , \quad \phi(r) = \frac{q''}{2kR}r^2$

با استفاده از جداسازی متغیرها : $\Psi = R(r).T(t) \quad \frac{d\Psi}{dt} = \frac{\alpha}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right)$

$$T'(t) + \alpha\lambda^2T(t) = 0 \Rightarrow T(t) = De^{-\alpha\lambda^2t}$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2rR = 0 \Rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2r^2R = 0 \Rightarrow v = 0$$

$$\Rightarrow R(r) = AJ_0(\lambda r) + BY_0(\lambda r)$$

$$\frac{d\Psi(0,t)}{dr}=0 \Rightarrow \frac{dR(0)}{dr}=0 \Rightarrow B=0$$

$$\frac{d\Psi(R,t)}{dr}=0 \Rightarrow y'_0(\lambda r)=0 \Rightarrow \lambda_n$$

$$\Rightarrow \Psi(r,t)=\sum_{n=0}^{\infty} q_n e^{-\alpha\lambda^2 t} J_0(\lambda r)$$

$$\Psi(r,0)=-\phi(r)-\Gamma(0)=-\frac{q''r^2}{2kR}-0=\frac{-q''r^2}{2kR} \Rightarrow a_n=\frac{\int_0^R r f(r) J_0(\lambda_n r)dr}{\int_0^R r J_0^2(\lambda_n r)dr}$$

$$y'_0(\lambda_n R)=0\ ,\frac{\lambda_n^2 R^2-v^2}{2\lambda_n^2}\lambda_v^2(\lambda_n R)=\frac{R^2}{2}J_0^2(\lambda_n R)$$

صورت کسر $=\frac{-q''}{2kR}\frac{R^3}{\lambda_n}y_1(\lambda_n R)=\frac{-q''R^2}{2k\lambda_n}\ y_1(\lambda_n R)$

$$\Rightarrow a_1=\frac{q''}{k\lambda_n}\frac{y_1(\lambda_n R)}{y_0^2(\lambda_n R)} \Rightarrow \Psi(r,t)=\frac{q''}{k}\sum_{n=1}^{\infty}\frac{y_1(\lambda_n R)}{y_0^2(\lambda_n R)}e^{-\lambda_n^2\alpha t}J_0(\lambda_n r)$$

$$T(r,t)=T_\infty-\frac{q''}{k}\sum_{n=1}^{\infty}\frac{y_1(\lambda_n R)}{y_0^2(\lambda_n R)}e^{-\lambda_n^2\alpha t}J_0(\lambda_n r)+\frac{q''}{2kR}r^2+\frac{2q''\alpha}{kR}t$$

مساله ۳۶-۳)

$$A(q_x-q_{x+dx})-hPdx(T-T_\infty)=0\ ,\ kA\frac{dT}{dx}-hP(T-T_\infty)=0$$

$$\theta=T-T_\infty\ ,BC\begin{cases}x=0 & \frac{dT}{dx}=0\\ x=X(t) & T=T_m\end{cases}$$

$$\frac{d^2\theta}{dx^2}-m^2\theta=0 \Rightarrow \theta=A\sinh(mx)+B\cosh(mx)$$

$$BC\ (1): A=0\ ,BC\ (2): \theta_m=B\cosh(mX(t)) \Rightarrow B=\frac{\theta_m}{\cosh(m.X(t))}$$

$$\Rightarrow \theta=\theta_m\frac{\cosh(mx)}{\cosh(m.X(t))}$$

$$q=kA\frac{d\theta}{dx}\Big|_{x=X(t)}=kA\theta_m\frac{\sinh(m.X(t))}{\cosh(m.X(t))}$$

$$\Rightarrow kA\theta_m\tanh(m.X(t))=\rho h_s A\frac{dX(t)}{dt}$$

$$\Rightarrow \frac{dX(t)}{\tanh(m.X(t))}=\frac{k_m\theta_m}{\rho h_s}dt \Rightarrow \frac{\cosh(m.X(t))}{\sinh(m.X(t))}dX(t)=\frac{k_m\theta_m}{\rho h_s}dt$$

$$\Rightarrow \frac{1}{m}Ln[\sinh(m.X(t))]=\frac{k_m\theta_m}{\rho h_s}t+D$$

$$at\ \ t=0 \Rightarrow x=L \Rightarrow D=\frac{1}{m}Ln(\sinh(mL))$$

$$\Rightarrow \frac{1}{m}(\text{Ln}[\sinh(m.X(t))]-\ln[\sinh(m.L)])=\frac{k\theta_m}{\rho h_s}t$$

$$\Rightarrow \text{Ln}\left[\frac{\sinh(m.X(t))}{\sinh(m.L)}\right]=\frac{mk\theta_m}{\rho h_s}t\ ,\ \beta=\frac{mk\theta_m}{\rho h_s}$$

$$\Rightarrow \sinh(m.X(t))=\sinh(m.L)e^{\beta t} \Rightarrow X(t)=\frac{1}{m}\sinh^{-1}[\sinh(m.L)e^{\beta t}]$$

نرخ ذوب شدن: $\dot{m}=\frac{dm}{dt}=\rho A.\frac{dX}{dt}=\frac{\rho A}{m}\frac{\beta\sinh(m.L)e^{\beta t}}{\sqrt{(\sinh(m.L)e^{\beta t})^2-1}}$

مساله ۳۷-۳)

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right)-hP(T-T_\infty)=0$$

$$\frac{d}{dx}\left(k\frac{d\theta}{dx}\right)-\frac{hP}{A}\theta=0\ ,\text{BC}\begin{cases}x=0: \frac{d\theta}{dx}=0\\ x=L: \theta=\theta_0\end{cases}$$

(a

اگر $k=k(T) \Rightarrow \frac{d}{dx}\left(k(T)\frac{d\theta}{dx}\right)-\frac{hP}{A}\theta=0$

$$\Rightarrow \left[\frac{dk}{dx}\cdot\frac{d\theta}{dx}+k(\theta)\cdot\frac{d^2\theta}{dx^2}\right]-\frac{hP}{A}\theta=0\ ,\frac{dk}{dx}=\frac{dk}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\Rightarrow \frac{dk}{d\theta}\cdot\left(\frac{d\theta}{dx}\right)^2+k(\theta)\cdot\frac{d^2\theta}{dx^2}-\frac{hP}{A}\theta=0$$

در $x=0: \frac{d\theta}{dx}=0 \Rightarrow k(\theta)\cdot\frac{d^2\theta}{dx^2}-\frac{hP}{A}\theta=0 \Rightarrow \frac{d^2\theta}{dx^2}=\frac{hP}{k(\theta)A}\theta$

$$\frac{dk}{d\theta}\cdot 2\frac{d\theta}{dx}\cdot\frac{d^2\theta}{dx^2}+k(\theta)\cdot\frac{d^3\theta}{dx^3}-\frac{hP}{A}\frac{d\theta}{dx}=0$$

در $x=0: \frac{d\theta}{dx}=0 \Rightarrow k(\theta)\cdot\frac{d^3\theta}{dx^3}=0 \Rightarrow \frac{d^3\theta}{dx^3}=0$

$$\frac{dk}{d\theta}\cdot 2\left(\frac{d^2\theta}{dx^2}\right)^2+\frac{dk}{d\theta}\cdot 2\frac{d\theta}{dx}\cdot\frac{d^3\theta}{dx^3}+k(\theta)\cdot\frac{d^4\theta}{dx^4}-\frac{hP}{A}\cdot\frac{d^2\theta}{dx^2}=0$$

$$\Rightarrow \frac{d^4\theta}{dx^4}=\left(\frac{hP}{Ak(\theta)}\right)^2\left(\theta-\frac{2}{k(\theta)}\cdot\frac{dk}{d\theta}\cdot\theta^2\right),$$

$$\Rightarrow \theta(x) = \theta(0) + \frac{1}{1!}\cdot\frac{d\theta}{dx}\Big|_{x=0} x + \frac{1}{2!}\cdot\frac{d^2\theta}{dx^2}\Big|_{x=0} x^2 + \frac{1}{3!}\cdot\frac{d^3\theta}{dx^3}\Big|_{x=0} x^3 + \frac{1}{4!}\cdot\frac{d^4\theta}{dx^4}\Big|_{x=0} x^4 + \cdots$$

$$\Rightarrow \theta(x) = \theta_0 + \frac{1}{2!}\cdot\left(\frac{hP}{Ak(\theta)}\right)\theta\cdot x^2 + \frac{1}{4!}\cdot\left(\frac{hP}{Ak(\theta)}\right)^2\left(\theta - \frac{2}{k(\theta)}\cdot\frac{dk}{d\theta}\cdot\theta^2\right)\cdot x^4$$

b)

اگر $k = k_0x^n \Longrightarrow \frac{d}{dx}\left(k_0x^n\frac{d\theta}{dx}\right) - \frac{hP}{A}\theta = 0$

$$\Rightarrow k_0nx^{n-1}\frac{d\theta}{dx} + k_0x^n\frac{d^2\theta}{dx^2} - \frac{hP}{A}\theta = 0, \text{(I)}$$

$$k_0n(n-1)x^{n-2}\frac{d\theta}{dx} + k_0nx^{n-1}\frac{d^2\theta}{dx^2} + k_0nx^{n-1}\frac{d^2\theta}{dx^2} + k_0x^n\frac{d^3\theta}{dx^3} - \frac{hP}{A}\frac{d\theta}{dx} = 0, \text{(II)}$$

$$k_0n(n-1)(n-2)x^{n-3}\frac{d\theta}{dx} + k_0n(n-1)x^{n-2}\frac{d^2\theta}{dx^2} + k_0n(n-1)x^{n-2}\frac{d^2\theta}{dx^2} + k_0nx^{n-1}\frac{d^3\theta}{dx^3} + k_0n(n-1)x^{n-2}\frac{d^2\theta}{dx^2} + k_0nx^{n-1}\frac{d^3\theta}{dx^3} + k_0x^n\frac{d^4\theta}{dx^4} + k_0nx^{n-1}\frac{d^3\theta}{dx^3} - \frac{hP}{A}\frac{d^2\theta}{dx^2} = 0 \text{ (III)}, \ldots$$

فصل چهارم

مسائل دو و سه بعدی پایا

مساله (۱-۴)

a) اطراف صفحه ایزوله است. δ کوچک است بنابراین مساله را در جهت‌های x و y در نظر می‌گیریم و در این حالت $h_3 = 0$ بنابراین:

$$q_xA_1|_x - q_xA_1|_{x+dx} + q_yA_2|_y - q_yA_2|_{y+dy} - h_1A_3(T-T_\infty) - h_2A_3(T-T_\infty) = 0$$

$A_1 = \delta dx \qquad A_2 = \delta dy \qquad A_3 = dx.dy$

$$\Rightarrow \frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} - \frac{(h_1+h_2)}{k\delta}(T-T_\infty) = 0$$

$$m^2=\frac{(h_1+h_2)}{k\delta}\quad ,T-T_\infty=\theta, BC\begin{cases}T(0,y)=T_0\\ T(\infty,y)=T_\infty\\ \frac{dT(x,0)}{dy}=0\\ \frac{dT\left(x,\frac{L}{2}\right)}{dy}=0\end{cases}$$

$$\Longrightarrow\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}-m^2\theta=0, BC\begin{cases}\theta(0,y)=\theta_0\\ \theta(\infty,y)=0\\ \frac{d\theta(x,0)}{dy}=0\\ \frac{d\theta\left(x,\frac{L}{2}\right)}{dy}=0\end{cases}$$

برای حل این مساله از روش جداسازی متغیرها استفاده می‌کنیم:

$$\theta(x,y)=X(x).Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}-m^2=\frac{-1}{Y}\frac{d^2Y}{dy^2}=+\lambda_n^2$$

جهت همگن مساله جهت y است و باید علامت λ_n^2 مخالف علامت جهت همگن در نظر گرفته شود بنابراین برای λ_n^2 علامت مثبت را در نظر می‌گیریم چون علامت y منفی است.

$$\frac{d^2Y}{dy^2}+\lambda_n^2Y=0\Longrightarrow Y=A_n\sin(\lambda_n y)+B_n\cos(\lambda_n y)$$

$$Y'(0)=0\Longrightarrow\frac{dY}{dy}=A_n\lambda_n\cos\lambda_n y-B_n\lambda_n\sin\lambda_n y\Longrightarrow A_n=0$$

$$\Longrightarrow Y=B_n\cos(\lambda_n y)$$

$$Y'\left(\frac{L}{2}\right)=0\Longrightarrow\frac{dY}{dy}=-B_n\lambda_n\sin\lambda_n\left(\frac{L}{2}\right)=0\Longrightarrow\sin\frac{\lambda_n L}{2}=0$$

$$\Longrightarrow\frac{\lambda_n L}{2}=n\pi\Longrightarrow\lambda_n=\frac{2n\pi}{L}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}-m^2=\lambda_n^2\Longrightarrow\frac{1}{X}\frac{d^2X}{dx^2}=\lambda_n^2+m^2\Longrightarrow\frac{d^2X}{dx^2}-(\lambda_n^2+m^2)X=0$$

$$X(x)=C_ne^{-\sqrt{\lambda_n^2+m^2}\,x}+D_ne^{\sqrt{\lambda_n^2+m^2}\,x}$$

$$X(\infty)=0\Longrightarrow D_n=0\Longrightarrow X(x)=C_ne^{-\sqrt{\lambda_n^2+m^2}\,x}$$

$$\theta(x,y)=\sum_{n=0}^{\infty}a_n\,e^{-\sqrt{\lambda_n^2+m^2}\,x}.\cos(\lambda_n y)$$

$$\theta(0,y)=\theta_0=\sum_{n=0}^{\infty}a_n\cos(\lambda_n y)\Longrightarrow a_n=\frac{\theta_0\int_0^{\frac{L}{2}}\cos(\lambda_n y)dy}{\int_0^{\frac{L}{2}}\cos^2(\lambda_n y)dy}$$

$$for\ n=0\Longrightarrow a_0=\theta_0$$

$$for\ n\neq0\Longrightarrow a_n=\frac{\theta_0\int_0^{\frac{L}{2}}\cos(\lambda_n y)dy}{\int_0^{\frac{L}{2}}\cos^2(\lambda_n y)dy}=\frac{\theta_0\left(\frac{1}{\lambda_n}\sin\left(\lambda_n\frac{L}{2}\right)\right)}{\frac{L}{2}\left(\frac{L}{2}+\frac{1}{2\lambda_n}\sin(\lambda_n L)\right)}=$$

$$\frac{2\theta_0\overbrace{\sin\left(\lambda_n\frac{L}{2}\right)}^{\sin n\pi=0}}{\lambda_n\left(\frac{L}{2}+\sin\left(\lambda_n\frac{L}{2}\right)\underbrace{\cos\left(\lambda_n\frac{L}{2}\right)}_{(-1)^n}\right)}=0$$

$$\Longrightarrow\theta(x,y)=\theta_0e^{-mx}\Longrightarrow\frac{T(x,y)-T_\infty}{T_0-T_\infty}=e^{-mx}$$

b) گرما از اطراف صفحه منتقل می‌شود

$$\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}-m^2\theta=0\ ,\ m^2=\frac{2h_3}{k\delta}\ ,\ \theta(x,y)=X(x).Y(y)$$

$$BC\begin{cases}\theta(0,y)=\theta_0\\ \theta(\infty,y)=0\\ \dfrac{d\theta(x,0)}{dy}=0\\ -k\dfrac{d\theta}{dy}\left(x,\frac{L}{2}\right)=h_3\theta\left(x,\frac{L}{2}\right)\end{cases}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}-m^2=-\frac{1}{Y}\frac{d^2Y}{dy^2}=+\lambda_n^2$$

$$\Rightarrow\frac{d^2Y}{dy^2}+\lambda_n^2 y=0\Rightarrow Y=A_n\sin(\lambda_n y)+B_n\cos(\lambda_n y)$$

$$BC\begin{cases}\dfrac{dY(0)}{dy}=0\Rightarrow A_n=0\Rightarrow Y=B_n\cos\lambda_n y\\ -k\dfrac{dY}{dy}\left(\frac{L}{2}\right)=h_3Y\left(\frac{L}{2}\right)\Rightarrow -k\left[-B_n\lambda_n y\sin\left(\lambda_n\frac{L}{2}\right)\right]=h_3B_n\cos\left(\lambda_n\frac{L}{2}\right)\end{cases}$$

$$\Rightarrow\tan\left(\lambda_n\cdot\frac{L}{2}\right)=\frac{h_3}{k\lambda_n}=\frac{h_3}{k\lambda_n}\times\frac{L}{L}=\frac{B_i}{\lambda_n L}\Rightarrow, n=0,1,2,\ldots$$

با حل این معادله λ_n به دست خواهد آمد

$$\frac{1}{x}\frac{d^2X}{dx^2}-m^2=+\lambda_n^2\Rightarrow X(x)=C_ne^{-\sqrt{\lambda_n^2+m^2}\,x}+D_ne^{\sqrt{\lambda_n^2+m^2}\,x}$$

$$X(\infty)=0\Rightarrow D_n=0\Rightarrow X(x)=C_ne^{-\sqrt{\lambda_n^2+m^2}x}$$

$$\Rightarrow\theta(x,y)=\sum_{n=0}^{\infty}a_n\,e^{-\sqrt{\lambda_n^2+m^2}x}.\cos\lambda_n y\ ,\ a_n=c_nB_n$$

$$X(0)=\theta_0\Rightarrow\theta(0,y)=\theta_0=\sum_{n=0}^{\infty}a_n.\cos\lambda_n y$$

$$\Rightarrow a_n=\frac{\int_0^{\frac{L}{2}}\theta_0\cos(\lambda_n y)dy}{\int_0^{\frac{L}{2}}\cos^2(\lambda_n y)dy}=\frac{\frac{\theta_0}{\lambda_n}\sin\left(\lambda_n\frac{L}{2}\right)}{\frac{L}{4}+\frac{1}{4\lambda_n}\sin(\lambda_n L)}$$

$$\Rightarrow\theta(x,y)=\sum_{n=0}^{\infty}\frac{\frac{\theta_0}{\lambda_n}\sin\left(\frac{\lambda_n L}{2}\right)}{\frac{L}{4}+\frac{1}{4\lambda_n}\sin(\lambda_n L)}.e^{-\sqrt{\lambda_n^2+m^2}\,x}.\cos\lambda_n y$$

$$\Rightarrow\frac{\theta(x,y)}{\theta_0}=\frac{T(x,y)-T_\infty}{T_0-T_\infty}=4\sum_{n=0}^{\infty}\frac{\sin\left(\frac{\lambda_n L}{2}\right)}{\lambda_n L+\sin(\lambda_n L)}e^{-\sqrt{\lambda_n^2+m^2}\,x}.\cos\lambda_n y$$

مساله ۲-۴)

$$q_x.A_1|_x-q_x.A_1|_{x+dx}+q_y.A_2|_y-q_y.A_2|_{y+dy}-h_1A_3(T-T_\infty)-$$
$$h_2A_3(T-T_\infty)+u'''.\delta.dx.dy=0$$

$$A_1=\delta dy\ ,A_2=\delta dx\ ,A_3=dy.dx$$

$$\Rightarrow\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}-m^2T+\frac{u'''}{k}=0\ ,m^2=\frac{(h_1+h_2)}{k\delta}$$

$$\Rightarrow\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}-m^2\theta+\frac{u'''}{k}=0$$

$$\theta(x,y)=\Psi(x,y)+\phi(x),BC\begin{cases}\dfrac{d\theta(0,y)}{dx}=0\\ \theta(L,y)=0\\ \dfrac{d\theta(x,0)}{dy}=0\\ \theta(x,l)=0\end{cases}$$

$$\Rightarrow\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}-m^2\Psi=0\ ,BC\begin{cases}\dfrac{d\Psi}{dx}(0,y)=0\\ \Psi(L,y)=0\\ \dfrac{d\Psi}{dy}(x,0)=0\\ \Psi(x,l)=-\phi(x)\end{cases}$$

$$\Rightarrow \frac{d^2\phi}{dx^2}-m^2\phi+\frac{u'''}{k}=0\ ,\ BC\begin{cases}\frac{d\phi}{dx}(0)=0\\ \phi(L)=0\end{cases}$$

$$\Rightarrow \phi(x)=c_1\sinh(mn)+c_2\cosh(mn)+\frac{u'''}{m^2k}\ ,\ \frac{d\phi}{dx}(0)=0\Rightarrow c_1=0$$

$$\phi(L)=0\Rightarrow c_2\cosh(mL)+\frac{u'''}{m^2k}=0\Rightarrow c_2=\frac{-\frac{u'''}{m^2k}}{\cosh(mL)}$$

$$\Rightarrow \phi(x)=\frac{u'''}{m^2k}\left[1-\frac{\cosh(mx)}{\cosh(mL)}\right]$$

$$\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}-m^2\Psi=0\ ,\ \Psi(x,y)=X(x).Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}+m^2=-\lambda^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda^2\Rightarrow X=B_n\cos(\lambda_n x)\ ,\ \lambda_n=\frac{(2n+1)\pi}{2L}\ ,n=0,1,2,\ldots$$

$$BC\begin{cases}\frac{dX(0)}{dx}=0\\ X(L)=0\end{cases}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=m^2+\lambda^2$$

$$\Rightarrow Y(y)=C_n\sinh\left(\sqrt{m^2+\lambda^2}.y\right)+D_n\cosh\left(\sqrt{m^2+\lambda^2}.y\right)$$

$$Y'(0)=0\Rightarrow Y(y)=D_n\cosh\left(\sqrt{m^2+\lambda^2}.y\right)$$

$$\Psi(x,y)=\sum_{n=0}^{\infty}a_n\cosh\left(\sqrt{m^2+\lambda_n^2}.y\right)\cos(\lambda_n x)$$

$$\Psi(x,l)=-\phi(x)$$

$$\sum_{n=0}^{\infty}a_n.\cosh\left(\sqrt{m^2+\lambda_n^2}.l\right)\cos(\lambda_n x)=-\frac{u'''m^2}{k}\left[1-\frac{\cosh(mx)}{\cosh(mL)}\right]$$

$$\Rightarrow a_n=\frac{\int_0^L\frac{u'''m^2}{k}\left[1-\frac{\cosh(mx)}{\cosh(mL)}\right]\cos(\lambda_n x)dx}{\cosh\left(\sqrt{m^2+\lambda_n^2}.l\right)\int_0^L\cos^2(\lambda_n x)dx}$$

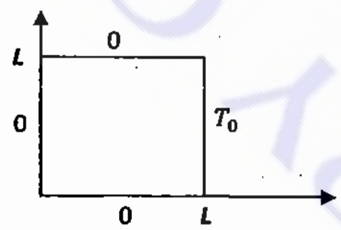
$$=\frac{2u'''m^2}{k\cosh\left(\sqrt{m^2+\lambda_n^2}.l\right)}.\frac{\int_0^L\left[1-\frac{\cosh(mx)}{\cosh(mL)}\right].\cos(\lambda_n x)dx}{\int_0^L\cos^2(\lambda_n x)dx}$$

$$\Rightarrow \theta(x,y)=\Psi(x,y)+\phi(x)$$

$$\Rightarrow \theta(x,y)=\sum_{n=0}^{\infty}a_n.\cosh\left(\sqrt{m^2+\lambda_n^2}\ .y\right).\cos(\lambda_n x)+\frac{u'''m^2}{k}\left[1-\frac{\cosh(mx)}{\cosh(mL)}\right]$$

مساله ۳-۴)

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,\ BC\begin{cases}T(0,y)=0\\ T(L,y)=T_0\\ T(x,L)=0\\ T(x,0)=0\end{cases}$$



$$T(x,y)=X(x).Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0\Rightarrow\frac{1}{X}\frac{d^2X}{dx^2}=\frac{-1}{Y}\frac{d^2Y}{dy^2}=+\lambda_n^2$$

$$\frac{d^2Y}{dy^2}+\lambda_n^2Y=0\Rightarrow Y(y)=A_n\sin(\lambda_n y)+B_n\cos(\lambda_n y)$$

$$BC\begin{cases}Y(L)=0\Rightarrow A_n\sin(\lambda_n L)=0\Rightarrow\lambda_n=\frac{n\pi}{L}\ ,n=0,1,2,\\ Y(0)=0\Rightarrow B_n=0\ldots\end{cases}$$

$$\frac{d^2X}{dx^2}-\lambda_n^2X=0\Rightarrow X(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$BC\begin{cases}X(0)=0\\ X(L)=T_0\end{cases}\Rightarrow \mathrm{D_n}=0\Rightarrow X(x)=\mathrm{C_n}\sinh(\lambda_n x)$$

$$T(x,y)=\sum_{\mathrm{n=0}}^{\infty}a_n.\sinh(\lambda_n x).\sin(\lambda_n y)\ ,\ a_n=A_n.C_n$$

$$T(L,y)=T_0=\sum_{\mathrm{n=0}}^{\infty}a_n.\sinh(\lambda_n L).\sin(\lambda_n y)$$

$$a_n=\frac{T_0\int_0^L\sin(\lambda_n y)dy}{\sinh(\lambda_n L)\underbrace{\int_0^L\sin^2(\lambda_n y)dy}_{\frac{L}{2}}}=\frac{2}{L\sinh(\lambda_n L)}.\ \underbrace{\int_0^L\sin(\lambda_n y)dy}_{\underbrace{\frac{-\cos\lambda_n y|_0^L}{\lambda_n}}_{\frac{-\cos\lambda_n L}{\lambda_n}+\frac{1}{\lambda_n}=\frac{1-\cos\lambda_n L}{\lambda_n}}}$$

مساله ۴-۴)

(a

$$(q_x-q_{x+dx})A_1+(q_y-q_{y+dy})A_2=0$$

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,\ \theta=T-T_\infty\Rightarrow\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}=0$$

$$BC\begin{cases}\theta(0,y)=0\\ \theta(l,y)=\theta_0=T_0-T_\infty\\ \frac{d\theta}{dy}(x,L)=0\\ \mathrm{k}\frac{d\theta}{dy}(x,0)=h\theta(x,0)\end{cases}$$

$$\theta(x,y)=X(x).Y(y)\Rightarrow\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=+\lambda_n^2$$

$$\Rightarrow\frac{d^2Y}{dy^2}+\lambda_n^2Y=0\Rightarrow Y(y)=A_n\sin(\lambda_n y)+B_n\cos(\lambda_n y)$$

$$BC\begin{cases}Y'(L)=0\Rightarrow\ A_n\lambda_n\cos(\lambda_n L)-B_n\lambda_n\sin(\lambda_n L)=0\\ \qquad\Rightarrow A_n=B_n\tan(\lambda_n L)\\ kY'(0)=hY(0)\Rightarrow kA_n\lambda_n=hB_n\ \Rightarrow\lambda_n kB_n\tan(\lambda_n L)=hB_n\\ \qquad\Rightarrow\tan(\lambda_n L)=\frac{h}{k\lambda_n}\end{cases}$$

با حل این معادله λ_n به دست خواهد آمد

$$\Rightarrow Y(y)=B_n[\tan(\lambda_n L).\sin(\lambda_n y)+\cos\lambda_n y]=$$

$$B_n\frac{\sin(\lambda_n L)\sin(\lambda_n y)+\cos(\lambda_n L).\cos(\lambda_n y)}{\cos(\lambda_n L)}=B_n\frac{\cos\lambda_n(L-y)}{\cos(\lambda_n L)}$$

$$\Rightarrow\frac{d^2X}{dx^2}-\lambda_n^2X=0\Rightarrow X(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$BC\begin{cases}X(0)=0\Rightarrow D_n=0\Rightarrow X(x)=C_n\sinh(\lambda_n x)\\ X(l)=\theta_0\end{cases}$$

$$\theta(x,y)=\sum_{\mathrm{n=0}}^{\infty}a_n.\sinh(\lambda_n x).\cos(\lambda_n(L-y))$$

$$a_n=C_nB_n\frac{1}{\cos(\lambda_n L)}$$

$$\theta(l,y)=\theta_0\Rightarrow\theta_0=\sum_{\mathrm{n=0}}^{\infty}a_n.\sinh(\lambda_n l).\cos(\lambda_n(L-y))$$

$$\Rightarrow a_n=\frac{\int_0^L\theta_0\cos[\lambda_n(L-y)]dy}{\int_0^L\cos^2[\lambda_n(L-y)]dy}\quad,\ \cos^2[\lambda_n(L-y)]=\frac{1+\cos(2\lambda_n(L-y))}{2}$$

$$\Rightarrow a_n=\frac{\theta_0\left(-\frac{1}{\lambda_n}\right)[\sin\lambda_n(L-L)-\sin\lambda_n L]}{\frac{1}{2}\left(L+\frac{\sin(2\lambda_n(L-L)-\sin2\lambda_n L)}{-\lambda_n}\right)}=\frac{-\theta_0\sin\lambda_n L}{\frac{-L\lambda_n}{2}-\sin(\lambda_n L)\cos(\lambda_n L)}$$

$$\frac{\theta}{\theta_0}=\sum_{\mathrm{n=0}}^{\infty}\frac{\sin\lambda_n L}{\frac{L\lambda_n}{2}+\sin(\lambda_n L)\cos(\lambda_n L)}.\sinh(\lambda_n x).\cos[\lambda_n(\mathrm{L}-\mathrm{y})]$$

b)

$$\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}=0\ ,BC\begin{cases}\theta(0,y)=0\\ \theta(l,y)=\theta_0\\ \frac{d\theta}{dy}(x,0)=0\\ -k\frac{d\theta}{dy}(x,L)=h\theta(x,L)\end{cases}$$

$$\theta(x,y)=X(x).Y(y)\Rightarrow\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=+\lambda_n^2$$

$$\frac{d^2Y}{dy^2}+\lambda_n^2Y=0\Rightarrow Y(y)=A_n\sin(\lambda_n y)+B_n\cos(\lambda_n y)$$

$$BC\begin{cases}Y'(0)=0\Rightarrow A_n=0\Rightarrow Y(y)=B_n\cos(\lambda_n y)\\ kY'(L)=hY(L)\Rightarrow kB_n\lambda_n\sin(\lambda_n L)=hB_n\cos(\lambda_n L)\\ \Rightarrow\tan(\lambda_n L)=\frac{h}{k\lambda_n}\ ,n=0,1,2,\dots\end{cases}$$

$$\frac{d^2X}{dx^2}-\lambda_n^2X=0\Rightarrow X(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$BC:X(0)=0\Rightarrow D_n=0\Rightarrow X(x)=C_n\sinh(\lambda_n x)$$

$$\theta(x,y)=\sum_{n=0}^{\infty}a_n.\sinh(\lambda_n x).\cos(\lambda_n y)$$

$$a_n=C_nB_n$$

$$\theta(l,y)=\theta_0\Rightarrow\theta_0=\sum_{n=0}^{\infty}a_n.\sinh(\lambda_n l).\cos(\lambda_n y)$$

$$\Rightarrow a_n=\frac{\theta_0\int_0^L\cos(\lambda_n y)dy}{\int_0^L\cos^2(\lambda_n y)dy}=\frac{\theta_0\left(\frac{1}{\lambda_n}\right)\sin(\lambda_n L)}{\int_0^L\frac{1+\cos(2\lambda_n y)}{2}dy}=\frac{\theta_0\sin\lambda_n L}{\left(\frac{L}{2}\right)\lambda_n+\sin(\lambda_n L).\cos(\lambda_n L)}$$

$$\Rightarrow\frac{\theta}{\theta_0}=\sum_{n=0}^{\infty}\left(\frac{\sin\lambda_n L}{\frac{L\lambda_n}{2}+\sin\lambda_n L.\cos\lambda_n L}\right).\sinh(\lambda_n x).\cos(\lambda_n y)$$

مساله ۵-۴)

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,BC\begin{cases}-k\frac{dT}{dx}(0,y)=h(T(0,y)-T_\infty)\\ -k\frac{dT}{dx}(l,y)=h(T(l,y)-T_\infty)-q''_2\\ -k\frac{dT}{dy}(x,0)=q''_1\\ T(x,L)=T_0\end{cases}$$

$$\theta=T-T_\infty\Rightarrow\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}=0\ ,BC\begin{cases}k\frac{d\theta}{dx}(0,y)=h\theta(0,y)\\ -k\frac{d\theta}{dx}(l,y)+q''_2=h\theta(l,y)\\ -k\frac{d\theta}{dy}(x,0)=q''_1\\ \theta(x,L)=\theta_0\end{cases}$$

$$\theta(x,y)=\theta_1(x,y)+\theta_2(x,y)+\theta_3(x,y)$$

$$\frac{d^2\theta_1}{dx^2}+\frac{d^2\theta_1}{dy^2}=0\ ,BC\begin{cases}+k\frac{d\theta_1}{dx}(0,y)=h\theta_1(0,y)\\ -k\frac{d\theta_1}{dx}(l,y)=h\theta_1(l,y)\\ -k\frac{d\theta_1}{dy}(x,0)=q''_1\\ \theta_1(x,L)=0\end{cases}$$

$$\frac{d^2\theta_2}{dx^2}+\frac{d^2\theta_2}{dy^2}=0\ ,\ BC\begin{cases}+k\frac{d\theta_2}{dx}(0,y)=h\theta_2(0,y)\\ -k\frac{d\theta_2}{dx}(l,y)=h\theta_2(l,y)\\ -k\frac{d\theta_2}{dy}(x,0)=0\\ \theta_2(x,L)=\theta_0\end{cases}$$

$$\frac{d^2\theta_3}{dx^2}+\frac{d^2\theta_3}{dy^2}=0\ ,\ BC\begin{cases}+k\frac{d\theta_3}{dx}(0,y)=h\theta_3(0,y)\\ -k\frac{d\theta_3}{dx}(l,y)=h\theta_3(l,y)-q''_2\\ -k\frac{d\theta_3}{dy}(x,0)=0\\ \theta_3(x,L)=0\end{cases}$$

حل این معادلات ساده به دانشجویان واگذار می‌شود.

مساله ۶-۴)

$$\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}=0\ ,\ \theta=T-T_\infty\ ,\ BC\begin{cases}\theta(0,y)=\theta_0\\ -k\frac{d\theta}{dx}(L,y)=h\theta(L,y)\\ \theta(x,0)=\theta_0\\ -k\frac{d\theta}{dy}(x,L)=h\theta(x,L)\end{cases}$$

$$\theta(x,y)=\theta_1(x,y)+\theta_2(x,y)$$

$$\frac{d^2\theta_1}{dx^2}+\frac{d^2\theta_1}{dy^2}=0\ ,\ BC\begin{cases}\theta_1(0,y)=0\\ -k\frac{d\theta_1}{dx}(L,y)=h\theta_1(L,y)\\ \theta_1(x,0)=\theta_0\\ -k\frac{d\theta_1}{dy}(x,L)=h\theta_1(x,L)\end{cases}$$

$$\frac{d^2\theta_2}{dx^2}+\frac{d^2\theta_2}{dy^2}=0\ ,\ BC\begin{cases}\theta_2(0,y)=\theta_0\\ -k\frac{d\theta_2(L,y)}{dx}=h\theta_2(L,y)\\ \theta_2(x,0)=0\,n\\ -k\frac{d\theta_2}{dy}(x,L)=h\theta_2(x,L)\end{cases}$$

$$\theta_1(x,y)=X_1(x).Y_1(y)\Rightarrow\frac{1}{X_1}\frac{d^2X_1}{dx^2}=-\frac{1}{Y_1}\frac{d^2Y_1}{dy^2}=-\lambda^2$$

$$\frac{d^2X_1}{dx^2}+\lambda_n^2X_1=0\Rightarrow X_1(x)=A_{1n}\sin\lambda_nx+B_{1n}\cos\lambda_nx$$

$$X_1(0)=0\Rightarrow B_{1n}=0\Rightarrow X_1(x)=A_{1n}\sin\lambda_nx\Rightarrow\frac{dX_1}{dx}=A_{1n}\lambda_n\cos\lambda_{nx}$$

$$\Rightarrow -k\frac{dX(L)}{dx}=hX(L)\Rightarrow -kA_{1n}\lambda_n\cos(\lambda_nL)=hA_{1n}\sin\lambda_nL\Rightarrow$$
$$\tan\lambda_nL=\frac{-k\lambda_n}{h}\ ,\ n=0,1,2,\dots$$

$$\frac{d^2Y_1}{dy^2}-\lambda_n^2Y_1=0\Rightarrow Y_1(y)=C_{1n}\sinh(\lambda_ny)+D_{1n}\cosh(\lambda_ny)$$

$$\frac{-k}{k}\frac{dY_1(L)}{dy}=Y_1(L)\Rightarrow\frac{-k\lambda_n}{h}[C_{1n}\cosh(\lambda_nL)+D_{1n}\sinh(\lambda_nL)]=$$
$$D_{1n}\cosh(\lambda_nL)+C_{1n}\sinh(\lambda_nL)$$

$$C_{1n}\left[\frac{-k\lambda_n}{h}\cosh(\lambda_nL)-\sinh(\lambda_nL)\right]=D_{1n}\left[\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)\right]$$

$$\Rightarrow C_{1n}=-D_{1n}\left[\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}\right]$$

$$Y(y)=D_{1n}\left[\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}\right](-\sinh(\lambda_ny)+\cosh(\lambda_ny))$$

$$\theta_1(x,y)=$$
$$\sum_{n=0}^{\infty}a_n\left\{-\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}\sinh(\lambda_ny)+\cosh(\lambda_ny)\right\}\sin(\lambda_nx)$$

$$\theta_1(x,0)=\theta_0\Rightarrow\theta_0=\sum_{n=0}^{\infty}a_n.\sin(\lambda_nx)\Rightarrow a_n=\frac{\theta_0\int_0^l\sin(\lambda_nx)dx}{\int_0^l\sin^2(\lambda_nx)dx}$$

$$=\frac{\theta_0\left(\frac{1}{\lambda_n}\right)(\cos\lambda_nL-1)}{\frac{1}{2}\left(L-\left(\frac{1}{2\lambda_n}\right)\sin(2\lambda_nL)\right)}=\frac{2\theta_0(\cos\lambda_nL-1)}{L\lambda_n-\sin(\lambda_nL)\cos(\lambda_nL)}$$

$$\frac{\theta_1}{\theta_0}=2\sum_{n=0}^{\infty}\left(\frac{\cos(\lambda_nL)-1}{L\lambda_n-\sin(\lambda_nL).\cos(\lambda_nL)}\right)\left(-\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}\sinh(\lambda_ny)+\right.$$
$$\left.\cosh(\lambda_ny)\right).\sin(\lambda_nx)$$

برای θ_2 می‌توان به صورت زیر نوشت:

$$\frac{\theta_2}{\theta_0}=2\sum_{n=0}^{\infty}\left(\frac{\cos(\lambda_nL)-1}{L\lambda_n-\sin(\lambda_nL).\cos(\lambda_nL)}\right)\left(-\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}.\sinh(\lambda_nx)+\right.$$
$$\left.\cosh(\lambda_nx)\right).\sin(\lambda_ny)$$

$$\frac{\theta}{\theta_0}=\frac{\theta_1}{\theta_0}+\frac{\theta_2}{\theta_0}=2\sum_{n=0}^{\infty}\left(\frac{\cos(\lambda_nL)-1}{L\lambda_n-\sin(\lambda_nL).\cos(\lambda_nL)}\right)\times$$
$$\left(-\frac{\cosh(\lambda_nL)+\frac{k\lambda_n}{h}\sinh(\lambda_nL)}{\sinh(\lambda_nL)+\frac{k\lambda_n}{h}\cosh(\lambda_nL)}(\sinh(\lambda_nx).\sin(\lambda_ny)+\sinh(\lambda_ny).\sin(\lambda_nx))\right.$$
$$\left.+\cosh(\lambda_nx).\sin(\lambda_ny)+\cosh(\lambda_ny).\sin(\lambda_nx)\right)$$

مساله ۸-۴)

$$(q_x-q_{x+dx})A_1+(q_y-q_{y+dy})A_2-(h_1+h_2)A_3(T-T_\infty)=0$$

$$A_1=t.dy=b\left(\frac{y}{L}\right)dy\ ,\ A_2=t.dx=b\left(\frac{y}{L}\right)dx\ ,\ A_3=dx.dy$$

$$q_x=-k\frac{\partial T}{\partial x}\Rightarrow k\left(\frac{b}{L}\right)\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)+k\left(\frac{b}{L}\right)\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right)-(h_1+h_2)\theta=0$$

$$\Rightarrow\frac{\partial^2\theta}{\partial x^2}+\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right)-\left(\frac{(h_1+h_2)l}{bky}\right)\theta=0\ ,\ m^2=\frac{(h_1+h_2)l}{bk}$$

$$\Rightarrow\frac{\partial^2\theta}{\partial x^2}+\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right)-\frac{m^2}{y}\theta=0\ ,\ BC\begin{cases}\frac{d\theta}{dy}(x,0)=0\\ \theta(x,L)=\theta_0\\ \frac{d\theta}{dx}(0,y)=0\\ -k\frac{d\theta}{dx}\left(\frac{l}{2},y\right)=h_3\theta\left(\frac{l}{2},y\right)\end{cases}$$

$$\theta(x,y)=X(x).Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{yY}\frac{d}{dy}\left(y\frac{dY}{dy}\right)-\frac{m^2}{y}=0$$

$$\frac{X''}{X}=-\frac{1}{yY}\frac{d}{dy}\left(y\frac{dY}{dy}\right)-\frac{m^2}{y}=-\lambda_n^2$$

$$X''+\lambda_n^2X=0\Rightarrow X(x)=A_n\sin\lambda_nx+B_n\cos\lambda_nx$$

$$X'(0)=0\Rightarrow A_n=0\Rightarrow X(x)=B_n\cos\lambda_nx$$

$$+kB_n\lambda_n\sin\left(\lambda_n\frac{l}{2}\right)=h_3B_n\cos\left(\lambda_n\frac{l}{2}\right)\Rightarrow\tan\left(\lambda_n\frac{l}{2}\right)=\frac{h_3}{k\lambda_n}.\frac{L}{L}$$

$\Rightarrow\tan\left(\lambda_n\frac{l}{2}\right)=\frac{h_3L}{k\lambda_nL}$ با حل این معادله λ_n به دست خواهد آمد

برای جهت y:

$$y^2Y'' + yY' - (m^2 + \lambda_n^2 y)yY = 0 \,,\, Y(y) = \sum_{n=0}^{\infty} a_n y^n$$

$$Y'(y) = \sum_{n=1}^{\infty} n a_n y^{n-1} \,,\, Y''(y) = \sum_{n=2}^{\infty} n(n-1) a_n y^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n y^n +$$

$$\sum_{n=1}^{\infty} n a_n y^n - m^2 \sum_{n=0}^{\infty} a_n y^{n+1} - \lambda_n^2 \sum_{n=0}^{\infty} a_n y^{n+2} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[n(n-1)a_n + n a_n - m^2 a_n - \lambda_n^2 a_n \right] y^n = 0$$

ادامه حل به خواننده واگذار می‌شود.

مساله ۹-۴)

$$q_x(\delta.dy) - q_{x+dx}(\delta.dy) + \rho\frac{\sqrt{2}}{2}U(\delta.dx)h^o + \rho\frac{\sqrt{2}}{2}U(\delta.dy)h^o -$$

$$\rho\frac{\sqrt{2}}{2}U(\delta.dy)\left(h^o + \frac{dh^o}{dx}dx\right) - \rho\frac{\sqrt{2}}{2}U(\delta.dx)\left(h^o + \frac{dh^o}{dy}dy\right) -$$

$$q_{x+dx}(\delta.dy) - q_{y+dy}(\delta.dx) - (h_1 + h_2)dx.dy(T - T_\infty) = 0$$

$$\Rightarrow -k\delta\frac{d^2T}{dx^2} - k\delta\frac{d^2T}{dy^2} - (h_1 + h_2)(T - T_\infty) - \rho u\delta c\frac{\sqrt{2}}{2}\frac{\partial T}{\partial y} - \rho u\delta c\frac{\sqrt{2}}{2}\frac{\partial T}{\partial x} =$$
$$0$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\rho uc}{k}\frac{\sqrt{2}}{2}\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\right) + \frac{h_1+h_2}{k\delta}(T - T_\infty) = 0$$

$$\theta = T - T_\infty \Rightarrow \frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{\rho uc}{k}\frac{\sqrt{2}}{2}\left(\frac{d\theta}{dx} + \frac{d\theta}{dy}\right) + \frac{h_1+h_2}{k\delta}\theta = 0$$

$$BC\begin{cases} \theta(0,y) = f(y) \\ \theta(x,0) = f(x) \\ \theta(L,y) = F(y) \\ \theta(x,L) = F(x) \end{cases}$$

با استفاده از قاعده جمع‌پذیری این مساله را به چهار مساله مجزا تقسیم می‌کنیم:

$$\frac{d^2\theta_1}{dx^2} + \frac{d^2\theta_1}{dy^2} + \frac{\rho uc}{k}\frac{\sqrt{2}}{2}\left(\frac{d\theta_1}{dx} + \frac{d\theta_1}{dy}\right) + \frac{h_1+h_2}{k\delta}\theta_1 = 0 \,,\, BC\begin{cases} \theta_1(0,y) = f(y) \\ \theta_1(x,0) = 0 \\ \theta_1(L,y) = 0 \\ \theta_1(x,L) = 0 \end{cases}$$

$$\theta_1(x,y) = X(x).Y(y) \Rightarrow X''Y + Y''X - \frac{\rho uc}{2k}(X'Y + Y'X) + \frac{h_1+h_2}{k\delta}XY = 0$$

$$\frac{X'' - \frac{\rho uc\sqrt{2}}{2k}X' + \frac{h_1+h_2}{k\delta}}{X} = -\frac{Y'' + \frac{\rho uc\sqrt{2}}{2k}Y'}{Y} = -\lambda_n^2$$

$$\Rightarrow Y'' + \frac{\rho uc\sqrt{2}}{2k}Y' + \lambda_n^2 Y = 0 \,,\, X'' - \frac{\rho uc\sqrt{2}}{2k}X' + \left(\frac{h_1+h_2}{k\delta} - \lambda_n^2\right)X = 0$$

$$Y'' + PY' + \lambda_n^2 Y = 0 \,,\, Y = S(y)U \Rightarrow Y' = S'U + SU'$$

$$Y'' = S''U + 2S'U' + SU''$$

$$\Rightarrow S''U + 2S'U' + SU'' + PS'U + PSU' + \lambda_n^2 SU = 0$$

$$\Rightarrow S''U + (2S' + PS)U' + \left(\lambda_n^2 S + PS' + S''\right)U = 0 \,**$$

$$2S' + PS = 0 \Rightarrow \frac{S'}{S} = -\frac{P}{2} \Rightarrow S = C_1 e^{-\frac{Py}{2}}$$

$S' = -\frac{P}{2}C_1 e^{-\frac{P}{2}y} \,,\, S'' = \frac{P^2}{4}C_1 e^{-\frac{P}{2}y} \Rightarrow$ با جایگزینی در **

$$C_1\exp\left(-\frac{P}{2}x\right)U'' + \lambda_n^2 C_1\exp\left(-\frac{P}{2}x\right) + P\left(-C_1\frac{P}{2}\exp\left(-\frac{P}{2}x\right)\right) +$$
$$\left(C_1\frac{P^2}{4}\exp(-Px)\right) = 0$$

$\Rightarrow U'' + \left(\lambda_n^2 - \frac{P^2}{4}\right)U = 0 \Rightarrow$ فرض: $\lambda_n^2 - \frac{P^2}{4} > 0 \Rightarrow$

$$U'' + \gamma^2 U = 0 \Rightarrow U = C_1\cos\gamma x + C_2\sin\gamma x$$

$$y(0) = 0 \Rightarrow C_1 = 0 \,,\, y(L) = 0 \Rightarrow \sin\gamma L = 0 \Rightarrow \gamma L = n\pi \,,\, n = 0,1,2,\ldots$$

$$\gamma = \frac{n\pi}{L} \Rightarrow \lambda_n^2 - \frac{P^2}{4} = \frac{n^2\pi^2}{L^2} \Rightarrow \lambda_n^2 = \frac{n^2\pi^2}{L^2} + \frac{P^2}{4}, n = 0,1,2,\dots$$

$$y(y) = S(y).U(y) \Rightarrow y(y) = C\exp\left(-\frac{Py}{2}\right).\sin\frac{n\pi y}{L}, P = \frac{\rho uc\sqrt{2}}{2}$$

$$X'' - PX' + (q - \lambda_n^2)X = 0 \Rightarrow \alpha^2 - P\alpha + (q - \lambda_n^2) = 0$$

$$\alpha_1 = \frac{P + \sqrt{P^2 - 4(q - \lambda_n^2)}}{2}, \alpha_2 = \frac{P - \sqrt{P^2 - 4(q - \lambda_n^2)}}{2}$$

$$X(x) = C_1\exp(\alpha_1 x) + C_2\exp(\alpha_2 x)$$

$$X(x) = 0 \Rightarrow C_1\exp(\alpha_1 L) + C_2\exp(\alpha_2 L) = 0$$

$$\Rightarrow C_2 = -C_1\exp(\alpha_1 - \alpha_2)L$$

$$\Rightarrow C_2 = -C_1\exp\left(\sqrt{P^2 - 4(q - \lambda_n^2)}L\right)$$

$$\Rightarrow \theta_1(x,y) = \sum_{n=1}^{\infty} C_n\left(1 - \exp\sqrt{P^2 - 4(q - \lambda_n^2)}.L\right)\exp\left(-\frac{P}{2}y\right)$$

$$\sin\gamma x.\exp\left(\frac{P + \sqrt{P^2 - 4(q - \lambda_n^2)}}{2}x\right)$$

$$\Rightarrow \theta_1(0,y) = g(y) =$$

$$\sum_{n=1}^{\infty} C_n\left(1 - \exp\sqrt{P^2 - 4(q - \lambda_n^2)}L\right)\exp\left(-\frac{P}{2}y\right).\sin\gamma y$$

$$\Rightarrow C_n\left(1 - \exp\sqrt{P^2 - 4(q - \lambda_n^2)}L\right) = \frac{\int_0^L g(y)\exp\left(-\frac{P}{2}y\right)\sin\gamma y dy}{\int_0^L \exp g(-Py)\sin^2\gamma y dy}$$

$$C_n = \frac{1}{1 - \exp\sqrt{P^2 - 4(q - \lambda_n^2)}L}\frac{\int_0^L g(y)\exp\left(-\frac{P}{2}y\right)\sin\gamma y dy}{\int_0^L \exp g(-Py)\sin^2\gamma y dy}$$

به همین ترتیب θ_2، θ_3 و θ_4 را به دست آورده و جواب نهایی حاصل جمع آنها خواهد بود:

$$\theta(x,y) = \theta_1(x,y) + \theta_2(x,y) + \theta_3(x,y) + \theta_4(x,y)$$

مساله ۱۰-۴)

$$t = b\left(\frac{x}{L}\right)^2$$

$P = b + 2l$ محیط جانبی

$A = \int_0^L b\left(\frac{x}{L}\right)^2 dx = \frac{bl}{3}$ سطح مقطع عرضی

$$(q_y - q_{y+dy}).\frac{bl}{3} + q''_2(2l.dy) + q''_1(b.dy) + h(2l + b)dy(T_\infty - T) = 0$$

$$\Rightarrow +k\frac{d^2T}{dy^2}dy.\frac{bl}{3} + (2q''_2 l + bq''_1)dy + (2l + b)(T_\infty - T)dy = 0$$

$$\Rightarrow \frac{d^2T}{dy^2} - \frac{3(2l+b)h}{kbl}(T - T_\infty) + \frac{3(2q''_2 l + bq''_1)}{kbl} = 0, \theta = T - T_\infty$$

$$\Rightarrow \frac{d^2\theta}{dy^2} - \underbrace{\frac{3(2l+b)}{kbl}}_{\alpha^2} + \underbrace{\frac{3(2q''_2 l + bq''_1)}{kbl}}_{\beta} = 0 \Rightarrow \frac{d^2\theta}{dy^2} - \alpha^2\theta + \beta = 0$$

$$\Rightarrow \theta = C_n\sinh(\alpha y) + D_n\cosh(\alpha y) + \frac{\beta}{\alpha^2}$$

$$BC\begin{cases}\frac{d\theta(0)}{dy} = \frac{+h_0}{k}(\theta - \theta_0)\\ \frac{d\theta(l)}{dy} = 0\end{cases}$$

$$\frac{d\theta}{dy} = \alpha\left(C_n\cosh(\alpha y) + D_n\sinh(\alpha y)\right)$$

$$\Rightarrow C_n = \frac{h_0}{k\alpha}\left(D_n + \frac{\beta}{\alpha^2} - \theta_0\right), D_n = \frac{\frac{h_0\beta}{k\alpha^3} + \frac{\theta_0 h_0}{k\alpha}}{\frac{h_0\beta}{k\alpha} + \tanh\alpha l}$$

$$\Rightarrow C_n = \left(\frac{\beta}{\alpha^2} - \theta_0\right)\frac{h_0}{k\alpha}\left(1 - \frac{\frac{h_0}{k\alpha}}{\frac{h_0}{k\alpha} + \tanh\alpha l}\right)$$

$$\theta(y)=\left(\frac{\beta}{\alpha^2}-\theta_0\right)\frac{h_0}{k\alpha}\left[\left(1-\frac{\frac{h_0}{k\alpha}}{\frac{h_0}{k\alpha}+\tanh\alpha l}\right)\sinh(\alpha y)+\frac{\cosh(\alpha y)}{\frac{h_0}{k\alpha}+\tanh\alpha l}+\frac{\beta}{\alpha^2}\right]$$

مساله ۱۱-۴)

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,BC\begin{cases}T(x,0)=T_0\\ T(x,l)=0\\ T(0,y)=T_0\\ T(l,y)=0\end{cases}\Rightarrow T(x,y)=T_1(x,y)+T_2(x,y)$$

$$\frac{d^2T_1}{dx^2}+\frac{d^2T_1}{dy^2}=0\ ,BC\begin{cases}T_1(x,0)=0\\ T_1(x,l)=0\\ T_1(0,y)=T_0\\ T_1(l,y)=0\end{cases}$$

$$\Rightarrow T_1(x,y)=X_1(x).Y_1(y)\Rightarrow\frac{1}{X_1}\frac{d^2X_1}{dx^2}=-\frac{1}{Y_1}\frac{d^2Y_1}{dy}=+\lambda_n^2$$

$$\Rightarrow Y_1(y)=A_{1n}\sin(\lambda_n y)+B_{1n}\cos(\lambda_n y)$$

$$Y_1(0)=0\Rightarrow B_{1n}=0\Rightarrow Y_1(y)=A_{1n}\sin(\lambda_n y)$$

$$Y_1(l)=0\Rightarrow\lambda_n l=n\pi\Rightarrow\lambda_n=\frac{n\pi}{l},n=0,1,2,\dots$$

$$,X_1(x)=C_{1n}\sinh(\lambda_n x)+D_{1n}\cosh(\lambda_n x)$$

$$X_1(l)=0\Rightarrow D_{1n}=-C_{1n}\tanh(\lambda_n l)\Rightarrow X_1(x)=C_{1n}\sinh(\lambda_n x)$$

$$-\tan(\lambda_n l).\cosh(\lambda_n x)=A\left(\frac{\cosh\lambda_n(l-x)}{\cosh(\lambda_n l)}\right)\Rightarrow$$

$$T_1(x,y)=\sum_{n=0}^{\infty}a_n\cosh\left(\lambda_n(l-x)\right)\sin\left(\lambda_n y\right)$$

$$\Rightarrow a_n=\frac{T_0\int_0^l\sin(\lambda_n y)dy}{\int_0^l\sin^2(\lambda_n y)dy}=\frac{2T_0((-1)^{n+1}+1)}{n\pi}$$

$$\Rightarrow\frac{T_1(x,y)}{T_0}=\sum_{n=0}^{\infty}\frac{2}{l}[(-1)^n-1]\cosh\lambda_n(l-x).\sin(\lambda_n y)$$

$$(-1)^n-1=\begin{cases}0 & n=2k\quad k=0,1,\dots\\ -2 & n=2k+1\quad k=0,1,\dots\end{cases}$$

$$\Rightarrow\frac{T_1(x,y)}{T_0}=\sum_{n=0}^{\infty}\frac{-4}{l}\cosh\left(\lambda_{2k+1}(l-x)\right)\sin(\lambda_{2k+1}y)$$

$$\lambda_{2k+1}=\frac{(2k+1)\pi}{l}$$

و برای $T_2(x,y)$:

$$\frac{T_2(x,y)}{T_0}=\sum_{k=0}^{\infty}\frac{-4}{l}\cosh\left(\lambda_{2k+1}(l-y)\right)\sin(\lambda_{2k+1}x)$$

$$\frac{T(x,y)}{T_0}=\frac{T_1+T_2}{T_0}=\frac{-4}{l}\sum_{n=0}^{\infty}\Big[\cosh\left(\lambda_{2k+1}(l-y)\right).\sin(\lambda_{2k+1}x)+$$
$$\cosh\left(\lambda_{2k+1}(l-x)\right).\sin(\lambda_{2k+1}y)\Big]$$

$$\frac{T\left(\frac{l}{2},\frac{l}{2}\right)}{T_0}=\frac{-4}{l}\sum_{k=0}^{\infty}\cosh\left(\lambda_{2k+1}\frac{l}{2}\right)\sin\left(\lambda_{2k+1}\frac{l}{2}\right)\times 2$$

$$,\lambda_{2k+1}\frac{l}{2}=(2k+1)\frac{\pi}{2}$$

$$\Rightarrow T\left(\frac{l}{2},\frac{l}{2}\right)=\frac{-8T_0}{l}\sum_{n=0}^{\infty}\cosh\left[(2k+1)\frac{\pi}{2}\right]$$

مساله ۱۲-۴)

در نقطه تماس این دو دیواره به دلیل وجود هوا مقاومت حرارتی وجود دارد. با توجه به این‌که این مقاومت حرارتی وجود دارد و ضخامت δ نیز کوچک است می‌توان فرض نمود که دیواره با ضخامت L از یک طرف ایزوله شده است.

از ضخامت δ در فرمولاسیون صرفنظر می‌کنیم.

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,\ BC\begin{cases}T(x,0)=0\\ \frac{dT(x,L)}{dy}=0\\ T(0,y)=T_0\\ T(\infty,y)=0\end{cases}$$

$$\Rightarrow T(x,y)=X(x).Y(y)\ \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=+\lambda_n^2$$

$$Y''+\lambda_n^2Y=0\Rightarrow Y=A_n\sin\lambda_n y+B_n\cos\lambda_n y\ ,\ Y(0)=0\Rightarrow B_n=0$$

$$X''-\lambda_n^2Y=0\Rightarrow X=C_ne^{\lambda_n x}+D_ne^{-\lambda_n x}$$

$$X(\infty)=0\Rightarrow C_n=0\quad ,\quad \frac{dY(L)}{dy}=0\Rightarrow \cos\lambda_n L=0\Rightarrow \lambda_n=\frac{(2n+1)\pi}{L}$$

$$n=0,1,2,3,\dots$$

$$T(x,y)=\sum_{n=0}^{\infty}a_n.\sin\lambda_n y.e^{-\lambda_n x}$$

$$T(0,y)=\sum_{n=0}^{\infty}a_n.\sin\lambda_n y=T_0\Rightarrow a_n=\frac{\int_0^l T_0\sin\lambda_n y\,dy}{\int_0^l\sin^2\lambda_n y\,dy}=\frac{2T_0}{\lambda_n L}(1-$$

$$\cos\lambda_n L)=\frac{2T_0}{\lambda_n L}\Rightarrow T(x,y)=\frac{4T_0}{\pi L}\sum_{n=0}^{\infty}\frac{e^{-\lambda_n x}}{2n+1}.\sin(\lambda_n y)$$

مساله ۱۳-۴)

$$\frac{d^2T}{dx^2}+\frac{d^2T}{dy^2}=0\ ,\ BC\begin{cases}T(0,y)=T_0\\ T(l,y)=0\\ T(x,0)=0\\ \frac{dT}{dy}(x,L)=0\end{cases}$$

$$T(x,y)=X(x).Y(y)$$

$$X''Y+Y''X=0\ ,\Rightarrow\frac{X''}{X}=-\frac{Y''}{Y}=+\lambda_n^2$$

$$Y''+\lambda_n^2Y=0\Rightarrow Y(y)=A_n\sin\lambda_n x+B_n\cos\lambda_n x$$

$$Y(0)=0\Rightarrow B_n=0\quad \frac{dY(L)}{dy}=0\Rightarrow\cos\lambda_n L=0\Rightarrow\lambda_n=\frac{(2n+1)\pi}{L}$$

$$n=0,1,2,\dots$$

$$X''-\lambda_n^2X=0\Rightarrow X(x)=C_ne^{\lambda_n x}+D_ne^{-\lambda_n x}$$

$$X(L)=0\Rightarrow C_ne^{\lambda_n l}+D_ne^{-\lambda_n l}=0\Rightarrow D_n=-C_ne^{2\lambda_n l}$$

$$T(x,y)=\sum_{n=0}^{\infty}a_n\left(e^{\lambda_n x}-e^{(2L-x)\lambda_n}\right)\sin\lambda_n y$$

$$T(0,y)=T_0=\sum_{n=0}^{\infty}a_n.e^{2L\lambda_n}.\sin\lambda_n y$$

$$\Rightarrow a_n\left(1-e^{2\lambda_n L}\right)=\frac{\int_0^L T_0\sin\lambda_n y\,dy}{\int_0^L\sin^2\lambda_n y\,dy}=\frac{2T_0}{L}\int_0^L\sin\lambda_n y\,dy=\frac{2T_0}{\lambda_n L}(1-$$

$$\cos\lambda_n L)$$

$$\Rightarrow a_n=\frac{2T_0}{\lambda_n L\left(1-e^{-2\lambda_n L}\right)}$$

$$\Rightarrow T(x,y)=\frac{2T_0}{L}\sum_{n=0}^{\infty}\frac{1}{\lambda_n\left(1-e^{-2\lambda_n L}\right)}\left(e^{\lambda_n x}-e^{\lambda_n(2L-x)}\right).\sin\lambda_n y$$

مساله ۱۴-۴)

(a

$$q_x.dy - q_{x+dx}.dy + q_y.dx - q_{y+dy}.dx + \rho V dy h^0 - \rho V dy\left(h^0 + \frac{dh^0}{dx}\right) = 0 \Rightarrow \frac{dq_y}{dy} + \frac{dq_x}{dx} + \rho V\frac{dh^0}{dx} = 0 \Rightarrow \frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} - \frac{\rho Vc}{k}\frac{dT}{dx} = 0$$

برای $x > 0$: $\frac{d^2T_1}{dx^2} + \frac{d^2T_1}{dy^2} - \frac{\rho Vc}{k}\frac{dT_1}{dx} = 0 \,,\, \frac{\rho Vc}{k} = 2S$

برای $x < 0$: $\frac{d^2T_2}{dx^2} + \frac{d^2T_2}{dy^2} - \frac{\rho Vc}{k}\frac{dT_2}{dx} = 0 \,,\, BC \begin{cases} (1)\quad \frac{dT_2(x,L)}{dy} = 0 \\ (2)\quad \frac{dT_2(x,0)}{dy} = 0 \\ (3)\quad T_2(-\infty, y) = 0 \\ (4)\quad \frac{dT_2(0,y)}{dx} = \frac{dT_1(0,y)}{dx} \\ (5)\quad T_2(0,y) = T_1(0,y) \\ (6)\quad q'' = k\frac{dT_1(x,L)}{dy} \\ (7)\quad \frac{dT_1(x,0)}{dy} = 0 \\ (8)\quad T_2(\infty, y) \propto x \end{cases}$

برای $x < 0$:

$$T_2(x,y) = X_2(x).Y_2(y) \Rightarrow X_2''Y_2 + X_2Y_2'' - 2SX'_2Y_2 = 0$$

$$\Rightarrow \frac{X_2'' - 2SX'_2}{X_2} = -\frac{Y_2''}{Y_2} = +\lambda_n^2$$

$$\Rightarrow Y_2'' + \lambda_n^2 Y_2 = 0 \Rightarrow Y_2(y) = A_{2n}\sin\lambda_n y + B_{2n}\cos\lambda_n y$$

$$(2): Y'_2(0) = 0 \Rightarrow A_{2n} = 0 \Rightarrow Y(y) = B_{2n}\cos\lambda_n y$$

$$(1): Y'_2(L) = 0 \Rightarrow \sin\lambda_n L = 0 \Rightarrow \lambda_n = \frac{n\pi}{L}, n = 1, 2, 3, \ldots$$

$$X''_2 - 2SX'_2 - \lambda_n^2 X_2 = 0 \,,\, X_2 = e^{rx} \Rightarrow r^2 - 2Sr - \lambda_n^2 = 0$$

$$r_{1,2} = \frac{2S \pm \sqrt{4S^2 + 4\lambda_n^2}}{2} \Rightarrow r_{1,2} = S \pm \sqrt{S^2 + \lambda_n^2}$$

$$\Rightarrow X(x) = C_{2n}e^{\left(S+\sqrt{S^2+\lambda_n^2}\right)x} + D_{2n}e^{-\left(S+\sqrt{S^2+\lambda_n^2}\right)x}$$

(3):

$$X(-\infty) = 0 \Rightarrow D_{2n} = 0 \Rightarrow T_2(x,y) = \sum_{n=0}^{\infty} a_n.e^{\left(S+\sqrt{S^2+\lambda_n^2}\right)x}.\cos\lambda_n y$$

برای $x > 0$ با در نظر گرفتن شرایط مرزی ناهمگن خواهیم داشت:

$$T_1(x,y) = \phi(x,y) + p(x) + q(y)$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} - 2s\frac{d\phi}{dx} + \frac{d^2p(x)}{dx^2} - 2S\frac{dp(x)}{dx} + \frac{d^2q(y)}{dy^2} = 0$$

$$\Rightarrow \begin{cases} \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} - 2S\frac{d\phi}{dx} = 0 \quad (*) \\ \frac{d^2p}{dx^2} - 2S\frac{dp}{dx} = -\frac{d^2q}{dy^2} \quad (**) \end{cases}$$

$$(**) \Rightarrow -\frac{d^2q}{dy^2} = C_1 \Rightarrow \frac{d^2q}{dy^2} = -C_1 \Rightarrow q(y) = -C_1\frac{y^2}{2} + C_2y + C_3$$

$$\frac{d^2p}{dx^2} - 2S\frac{dp}{dx} = C_1 \Rightarrow p(x) = A_1 + A_2e^{2Sx} - \frac{C_1x}{2S}$$

به صورت اختیاری $A_1 = 0 \Rightarrow p(x) = A_2e^{2Sx} - \frac{C_1x}{2S}$

BC. 8: $T_1(\infty, y) \propto x \Rightarrow \phi(\infty, y) + p(\infty) + q(y) \propto x$

$$\Rightarrow p(\infty) \propto x \Rightarrow A_2 = 0 \Rightarrow p(x) = -\frac{C_1 x}{2S}$$

$$\text{BC. 7: } \frac{dT_1(x,y)}{dy} = 0 \Rightarrow \frac{d\phi(x,y)}{dy} + \frac{dq(0)}{dy} = 0 \Rightarrow \begin{cases} \frac{d\phi(x,0)}{dy} = 0 \\ \frac{dq(0)}{dy} = 0 \end{cases}$$

$$\text{BC. 6: } \frac{dT_1(x,L)}{dy} \Rightarrow \frac{q''}{k} = \frac{d\phi(x,L)}{dy} + \frac{dq(L)}{dy} \Rightarrow \begin{cases} \frac{d\phi(x,L)}{dy} = 0 \\ \frac{dq(L)}{dy} = \frac{q''}{k} \end{cases}$$

$$\Rightarrow q(y) = -C_1\frac{y^2}{2} + C_2 y + C_3 \text{ , } BC \begin{cases} \frac{dq(0)}{dy} = 0 \\ \frac{dq(L)}{dy} = \frac{q''}{k} \end{cases}$$

فرض: $C_3 = 0$

$$\frac{dq(0)}{dy} = 0 \Rightarrow C_2 = 0 \text{ , } \frac{dq(L)}{dy} = \frac{q''}{k} \Rightarrow -C_1 L = \frac{q''}{k} \Rightarrow C_1 = -\frac{q''}{kL}$$

$$\Rightarrow q(y) = \frac{q''}{2kL}y^2$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} - 2S\frac{d\phi}{dx} = 0 \text{ , } BC \begin{cases} \frac{d\phi(x,0)}{dy} = 0 \\ \frac{d\phi(x,L)}{dy} = 0 \end{cases}$$

$$\phi(x,y) = f(x).g(y)$$

$$\frac{f'' - 2sf'}{f} = -\frac{g''}{g} = +\lambda_n^2 \Rightarrow$$

$$\begin{cases} g'' + \lambda_n^2 g = 0 \Rightarrow g(y) = A\sin\lambda_n y + B\cos\lambda_n y \\ f'' - 2Sf' - \lambda_n^2 f = 0 \Rightarrow by\ X = e^{rx} \Rightarrow r^2 - 2Sr - \lambda_n^2 = 0 \end{cases}$$

$$\Rightarrow r_{1,2} = \left(S \pm \sqrt{S^2 + \lambda_n^2}\right) \Rightarrow f(x) = Ce^{\left(S+\sqrt{S^2+\lambda_n^2}\right)x} + De^{\left(S-\sqrt{S^2+\lambda_n^2}\right)x}$$

$$\frac{d\phi(x,0)}{dy} = 0 \Rightarrow \frac{dg(0)}{dy} = 0 \Rightarrow A = 0$$

$$\frac{d\phi(x,L)}{dy} = 0 \Rightarrow \frac{dg(L)}{dy} = 0 \Rightarrow \sin\lambda_n L = 0 \Rightarrow \lambda_n = \frac{n\pi}{L}, n = 0,1,2,\dots$$

$$T_1(\infty,y) \propto x \Rightarrow \phi(\infty,y) \propto x \Rightarrow C = 0$$

$$\Rightarrow T_1(x,y) = \underbrace{\sum_{n=0}^{\infty} b_n e^{\left(S-\sqrt{S^2+\lambda_n^2}\right)x}.\cos\lambda_n y}_{\phi(x,y)} + \underbrace{\frac{q''}{2kL}}_{q(y)} + \underbrace{\frac{q''}{2kLS}x}_{p(x)}$$

$$\text{BC 5: } T_2(0,y) = T_1(0,y) \Rightarrow \sum_{n=0}^{\infty} b_n\cos\lambda_n y + \frac{q''y^2}{2kL} = \sum_{n=0}^{\infty} a_n.\cos\lambda_n y$$

$$\Rightarrow \sum_{n=0}^{\infty}(a_n - b_n)\cos\lambda_n y = \frac{q''y^2}{2kL}$$

$$\Rightarrow a_n - b_n = \frac{\frac{q''}{2kL}\int_{-L}^{+L} y^2\cos\lambda_n y\, dy}{\int_{-L}^{+L}\cos^2\lambda_n y\, dy}$$

$$\int_{-L}^{+L} y^2\cos\lambda_n y\, dy = \left(\frac{y^2}{\lambda_n}\sin\lambda_n y - \frac{2}{\lambda_n^3}\sin\lambda_n y + \frac{2y}{\lambda_n^2}\cos\lambda_n y\right)_{-L}^{+L} =$$

$$\frac{4L}{\lambda_n^2}\cos\lambda_n L = \frac{4L(-1)^n}{\lambda_n^2}$$

$$\int_{-L}^{+L}\cos^2\lambda_n y\,dy=\left(\frac{y}{2}+\frac{\sin 2\lambda_n y}{4\lambda_n}\right)_{-L}^{+L}=L$$

$$a_n-b_n=\frac{\frac{q''}{2kL}\cdot\frac{4L(-1)^n}{\lambda_n^2}}{L}=2\frac{q''(-1)^n}{Lk\lambda_n^2}$$

BC.4: $\frac{dT_2(0,y)}{dx}=\frac{dT_1(0,y)}{dx}$

$$\sum_{n=0}^{\infty}a_n\left(S+\sqrt{\lambda_n^2+S^2}\right)\cos\lambda_n y=\sum_{n=0}^{\infty}b_n\left(S-\sqrt{\lambda_n^2+S^2}\right)\cos\lambda_n y+\frac{q''}{2kLS}$$

$$\Rightarrow\sum_{n=0}^{\infty}\left[\left(S+\sqrt{\lambda_n^2+S^2}\right)a_n-\left(S-\sqrt{\lambda_n^2+S^2}\right)b_n\right]\cos\lambda_n y=\frac{q''}{2kLS}$$

$$\Rightarrow\left(S+\sqrt{\lambda_n^2+S^2}\right)a_n-\left(S-\sqrt{\lambda_n^2+S^2}\right)b_n=\frac{\frac{q''}{2kLS}\int_{-L}^{+L}\cos\lambda_n y\,dy}{\int_{-L}^{+L}\cos^2\lambda_n y\,dy}$$

$$\int_{-L}^{+L}\cos\lambda_n y\,dy=\frac{1}{\lambda_n}(\sin\lambda_n L-\sin\lambda_n L)=0$$

$$\Rightarrow a_n=\frac{S-\sqrt{\lambda_n^2+S^2}}{S+\sqrt{\lambda_n^2+S^2}}b_n\ \text{،}\ a_n-b_n=\frac{2q''(-1)^n}{Lk\lambda_n^2}$$

$$\Rightarrow\frac{S-\sqrt{S^2+\lambda_n^2}}{S+\sqrt{S^2+\lambda_n^2}}b_n-b_n=\frac{2q''(-1)^n}{Lk\lambda_n^2}\Rightarrow b_n\left(\frac{S-\sqrt{S^2+\lambda_n^2}}{S+\sqrt{S^2+\lambda_n^2}}-1\right)=\frac{2q''(-1)^n}{Lk\lambda_n^2}$$

$$\Rightarrow b_n\left(\frac{S-\sqrt{S^2+\lambda_n^2}}{S+\sqrt{S^2+\lambda_n^2}}-1\right)=\frac{2q''(-1)^n}{Lk\lambda_n^2}\Rightarrow b_n=\frac{-q''(-1)^n\left(S+\sqrt{S^2+\lambda_n^2}\right)}{Lk\lambda_n^2\sqrt{S^2+\lambda_n^2}}$$

$$\Rightarrow a_n=\frac{-q''(-1)^n\left(S-\sqrt{S^2+\lambda_n^2}\right)}{Lk\lambda_n^2\sqrt{S^2+\lambda_n^2}}$$

$$\Rightarrow T_2(x,y)=\sum_{n=0}^{\infty}\frac{-q''(-1)^n\left(S+\sqrt{S^2+\lambda_n^2}\right)}{Lk\lambda_n^2\sqrt{S^2+\lambda_n^2}}.e^{\left(S+\sqrt{S^2+\lambda_n^2}\right)x}.\cos\lambda_n y$$

$$\lambda_n=\frac{n\pi}{L},S=\frac{\rho Vc}{2k}\Rightarrow\left(S+\sqrt{S^2+\lambda_n^2}\right)x=\left(\frac{\rho Vc}{2k}+\sqrt{\left(\frac{\rho Vc}{2k}\right)^2+\left(\frac{n\pi}{L}\right)^2}\right)x$$

$$=\left(\frac{\rho VcL}{2k}+\sqrt{\left(\frac{\rho Vc}{2k}\right)^2+(n\pi)^2}\right)\left(\frac{x}{L}\right)=\left(\frac{VL}{2a}+\sqrt{\left(\frac{VL}{2a}\right)^2+(n\pi)^2}\right)\left(\frac{x}{l}\right)$$

$$\left(\frac{Vl}{a}\right)=p\,,\frac{1}{\xi}=\frac{p}{\frac{x}{L}}\,,\eta=\frac{y}{L}\Rightarrow\cos(\lambda_n y)=\cos(n\pi\eta)$$

$$\Rightarrow\left(S+\sqrt{S^2+\lambda_n^2}\right)x=\left(\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+(n\pi)^2}\right)(p\xi)=\frac{i}{2}\left(1+\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}\right)p^2\xi$$

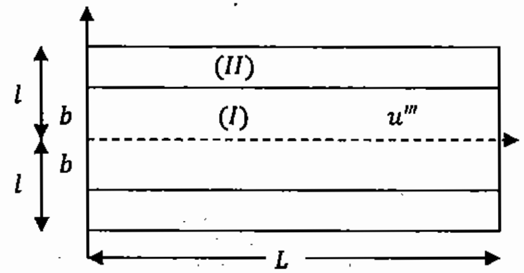
$$\Rightarrow T_2(x,y)=$$

$$\sum_{n=0}^{\infty}\frac{-q''(-1)^n\left(\frac{p}{2}\right)\left(1-\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}\right)p}{\frac{(n\pi)^2pk}{2L^2}\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}}\exp\left(\frac{1}{2}\left(1+\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}\right)p^2\xi\right).\cos n\pi\ \eta$$

$$\Rightarrow T_1(x,y)=$$

$$\sum_{n=0}^{\infty}\frac{-q''(-1)^n\left(\frac{p}{2}\right)\left(1+\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}\right)p}{\frac{(n\pi)^2pk}{2l^2}\sqrt{1+\left(\frac{2n\pi}{p}\right)^2}}\exp\left(\frac{1}{2}\left(1-\sqrt{1-\left(\frac{2n\pi}{p}\right)^2}\right)p^2\xi\right).\cos n\pi\ \eta+\frac{q''l}{2k}\eta^2+\frac{\xi}{p}$$

مساله ۱۵-۴)



به دلیل تقارن نصف شکل را در نظر می‌گیریم

برای بخش (I):

$$u'''.\delta.dx.dy + q_y.dx.\delta - q_{y+dy}.dx.\delta + \rho U\delta dy.h^0 - \rho U\delta dy\left(h^0 + \frac{dh^0}{dx}dx\right) - (h_1+h_2)dx.dy(T-T_\infty) = 0\ ,\ q_y = -k\frac{dT}{dy}\ ,\ h^0 = CT$$

$$-k\delta\frac{d^2T_1}{dy^2} + \rho U\delta c\frac{dT_1}{dx} + (h_1+h_2)(T-T_\infty) - u'''.\delta = 0$$

$$\Longrightarrow \frac{d^2T_1}{dy^2} - \frac{\rho Uc}{k}\frac{dT_1}{dx} - \frac{(h_1+h_2)}{k\delta}(T-T_\infty) + \frac{u'''}{k} = 0$$

برای بخش (II):

$$\frac{d^2T_2}{dy^2} - \frac{\rho Uc}{k}\frac{dT_2}{dx} - \frac{(h_1+h_2)}{k\delta}(T_2-T_\infty) = 0$$

$$\theta_1 = T_1 - T_\infty\ ,\ \theta_2 = T_2 - T_\infty$$

$$(I) \Longrightarrow \frac{d^2\theta_1}{dy^2} - \frac{U}{\propto}\frac{d\theta_1}{dx} - H\theta_1 + \frac{u'''}{k} = 0\ ,\ H = \frac{(h_1+h_2)}{k\delta}$$

$$(II) \Longrightarrow \frac{d^2\theta_2}{dy^2} - \frac{U}{\propto}\frac{d\theta_2}{dx} - H = 0$$

$$BC\begin{cases} (1)\ \ \dfrac{dT_1(x,0)}{dy} = 0 \Longrightarrow \dfrac{d\theta_1(x,0)}{dy} = 0 \\ (2)\ \ \dfrac{dT_1(x,b)}{dy} = \dfrac{dT_2(x,b)}{dy} \Longrightarrow \dfrac{d\theta_1(x,b)}{dy} = \dfrac{d\theta_2(x,b)}{dy} \\ (3)\ \ T_1(x,b) = T_2(x,b) \Longrightarrow \theta_1(x,b) = \theta_2(x,b) \\ (4)\ \ T_1(0,y) = T_0 \Longrightarrow \theta_1(0,y) = T_0 - T_\infty \\ (5)\ \ T_2(0,y) = T_0 \Longrightarrow \theta_2(0,y) = T_0 - T_\infty \\ (6)\ \ \dfrac{dT_2(x,l)}{dy} = 0 \Longrightarrow \dfrac{d\theta_2(x,l)}{dy} = 0 \end{cases}$$

حل معادله I:

$$\theta_1(x,y) = \Psi(x,y) + \phi(x) \overset{(I)}{\Longrightarrow} \frac{d^2\Psi}{dy^2} - \frac{u}{\propto}\frac{d\Psi}{dx} - H\Psi - \frac{U}{\propto}\frac{d\phi}{dx}\frac{u'''}{k} = 0$$

$$\Longrightarrow \begin{cases} \dfrac{d^2\Psi}{dy^2} - \dfrac{U}{\propto}\dfrac{d\Psi}{dy} + H\Psi = 0 \quad (**) \\ -\dfrac{U}{\propto}\dfrac{d\phi}{dx} = -\dfrac{u'''}{k} \quad (*) \end{cases}$$

$(*) \Longrightarrow \phi(x) = \frac{\propto u''' x}{Uk} + c_1$ فرض: $c_1 = 0 \Longrightarrow \phi(x) = \frac{\propto u''' x}{Uk}$

$$(**) \overset{(I)}{\Longrightarrow} \frac{d^2\Psi}{dy^2} - \frac{U}{\propto}\frac{d\Psi}{dy} + H\Psi = 0$$

$$\Psi(x,y) = X(x).Y(y) \Longrightarrow Y''X - 2SX'Y - HXY = 0$$

$$\Longrightarrow \frac{Y''}{Y} = \frac{2SX'+HX}{X} = -\lambda_n^2$$

$$Y'' + \lambda_n^2 Y = 0 \Longrightarrow Y(x) = A_n\sin\lambda_n y + B_n\cos\lambda_n y$$

$$2SX' + (H+\lambda_n^2)X = 0 \Longrightarrow X = C_n e^{-\frac{(H+\lambda_n^2)}{2S}x}$$

$$\frac{d\theta_1(x,0)}{dy} = 0 \Longrightarrow \frac{d\Psi(x,0)}{dy} = 0 \Longrightarrow \frac{dY(0)}{dy} = 0 \Longrightarrow A_n = 0$$

$$\Longrightarrow \theta_1(x,y) = \sum a_n\cos\lambda_n y.e^{-\frac{(H+\lambda_n^2)}{2S}x} + \frac{\propto u''' x}{Uk}$$

حل معادله II:

$$\theta_2(x,y) = X(x).Y(y) \Rightarrow Y''X - 2SX'Y - HXY = 0$$

$$\Rightarrow \frac{Y''}{Y} = \frac{2SX'+HX}{X} = -\lambda_n^2 \Rightarrow \begin{cases} Y'' + \lambda_n^2 Y = 0 \\ 2SX' + \left(H + \lambda_n^2 Y\right)X = 0 \end{cases}$$

$$\Rightarrow Y(y) = D_n \sin\lambda_n y + E_n \cos\lambda_n y\ ,\ X(x) = G_n e^{-\frac{\left(H+\lambda_n^2\right)}{2S}x}$$

$$\frac{dY(l)}{dy} = 0 \Rightarrow D_n \cos\lambda_n l - E_n \sin\lambda_n l = 0 \Rightarrow D_n = E_n \tan\lambda_n l$$

$$\Rightarrow \theta_2(x,y) = \sum_{n=0}^{\infty} b_n.\left(\tan\lambda_n l.\sin\lambda_n y + \cos\lambda_n y\right) e^{-\frac{\left(H+\lambda_n^2\right)}{2S}x}$$

$$\theta_0 = \theta_1(x,y) = \sum_{n=0}^{\infty} a_n \cos\lambda_n y \Rightarrow a_n = \frac{\theta_0 \int_0^b \cos\lambda_n y\, dy}{\int_0^b \cos^2\lambda_n y\, dy} =$$

$$\frac{\theta_0 \frac{1}{\lambda_n}\left(\sin\lambda_n b\right)}{\frac{1}{2}\left(b + \frac{1}{2\lambda_n}\sin\lambda_n b\right)} = \frac{4\theta_0 \sin\lambda_n b}{2b\lambda_n + \sin\lambda_n b}$$

$$\theta_0 = \theta_2(0,y) = \sum_{n=0}^{\infty} b_n \left(\tan\lambda_n l.\sin\lambda_n y + \cos\lambda_n y\right)$$

$$\Rightarrow b_n \tan\lambda_n l = \frac{\theta_0 \int_b^l \sin\lambda_n y\, dy}{\int_b^l \sin^2\lambda_n y\, dy} == \frac{4\theta_0\left(\sin\lambda_n l - \sin\lambda_n b\right)}{2(l-b)\lambda_n + \left(\sin\lambda_n l - \sin\lambda_n b\right)}$$

$$\Rightarrow \theta_1(x,y) = \sum_{n=0}^{\infty} \frac{4\theta_0 \sin\lambda_n b}{2b\lambda_n + \sin\lambda_n b}.\cos\lambda_n.e^{-\frac{\left(H+\lambda_n^2\right)}{2S}x} + \frac{\propto u''' x}{Uk}$$

و

$$\theta_2(x,y) = \sum_{n=0}^{\infty} \frac{4\theta_0\left(\sin\lambda_n l - \sin\lambda_n b\right)}{2(l-b)\lambda_n + \left(\sin\lambda_n l - \sin\lambda_n b\right).\tan\lambda_n l}.\left(\tan\lambda_n l.\sin\lambda_n y + \cos\lambda_n y\right) e^{-\frac{\left(H+\lambda_n^2\right)}{2S}x}$$

$$\theta_1(x,b) = \theta_2(x,b) \Rightarrow X_1 Y_1(b) = X_2 Y_2(b)$$

$$\frac{d\theta_1(x,b)}{dy} = \frac{d\theta_2(x,b)}{dy} \Rightarrow X_1 \frac{dY_1(b)}{dy} = X_2 \frac{dY_2(b)}{dy}$$

$$\frac{Y'_1(b)}{Y_1(b)} = \frac{Y'_2(b)}{Y_2(b)} \Rightarrow \frac{\lambda_n \tan\lambda_n l.\cos\lambda_n b - \lambda_n \sin\lambda_n b}{\tan\lambda_n l.\sin\lambda_n b + \cos\lambda_n b} = \frac{-\sin\lambda_n b}{\cos\lambda_n b} \Rightarrow$$

با حل این معادله λ_n به دست خواهد آمد

مساله ۱۶-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{d^2T}{r^2 d\phi^2} = 0 \Rightarrow r^2\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{d^2T}{d\phi^2} = 0$$

$$BC\begin{cases} T(R_i,\varphi) = T_i(\varphi) \\ T(R_0,\varphi) = T_0(\varphi) \end{cases} \Rightarrow \text{جهت } r \text{ ناهمگن است}$$

$$\begin{cases} T(r,\varphi) = T(r,\varphi+2\pi) \\ \frac{dT(r,\varphi)}{rd\varphi} = \frac{dT(r,\varphi+2\pi)}{rd\varphi} \end{cases} \Rightarrow \text{جهت } \varphi \text{ ناهمگن است}$$

$$\Rightarrow T(r,\varphi) = T_1(r,\varphi) + T_2(r,\varphi)$$

$$r^2\frac{d}{dr}\left(r\frac{dT_1}{dr}\right) + \frac{d^2T}{d\varphi^2} = 0\ ,\ BC\begin{cases} T_1(R_i,\varphi) = T_i(\varphi) \\ T_1(R_0,\varphi) = 0 \\ T_1(r,\varphi) = T_1(r,\varphi+2\pi) \\ \frac{dT_1(r,\varphi)}{rd\varphi} = \frac{T_1(r,\varphi+2\pi)}{rd\varphi} \end{cases}$$

و

$$r^2\frac{d}{dr}\left(r\frac{dT_2}{dr}\right) + \frac{d^2T_2}{d\varphi^2} = 0\ ,\ BC\begin{cases} T_2(R_i,\varphi) = 0 \\ T_2(R_0,\varphi) = T_0(\varphi) \\ T_2(r,\varphi) = T_2(r,\varphi+2\pi) \\ \frac{dT_2(r,\varphi)}{rd\varphi} = \frac{dT_2(r,\varphi+2\pi)}{rd\varphi} \end{cases}$$

حل $T_1(r,\varphi)$ با استفاده از روش جداسازی متغیرها:

$$T_1(r,\varphi) = R_1(r).\phi_1(\varphi) \Rightarrow \frac{r}{R_1}\frac{d}{dr}\left(r\frac{dR_1}{dr}\right) = \frac{-1}{\phi_1}\frac{d^2\phi_1}{d\varphi^2} = +\lambda_n^2$$

$$\Rightarrow \phi''_1 + \lambda_n^2\phi_1 = 0 \Rightarrow \phi_1(\varphi) = A_{1n}\sin\lambda_n\varphi + B_{1n}\cos\lambda_n\varphi$$

$$\frac{d\phi_1(\varphi)}{d\varphi} = \frac{d\phi_1(\varphi+2\pi)}{d\varphi} \Rightarrow \lambda_n = n\ ,\ n = 0,1,2,\ldots$$

$$\frac{d}{dr}\left(r\frac{dR_1}{dr}\right)-\lambda_n^2\frac{R_1}{r}=0\Rightarrow r^2\frac{d^2R_1}{dr}+r\frac{dR_1}{dr}-\lambda_n^2R_1=0$$

$$\Rightarrow R_1(r)=C_{1n}.r^{\lambda_n}+D_{1n}.r^{-\lambda_n}=C_{1n}.r^n+D_{1n}.r^{-n}$$

$$R_1(R_0)=0\Rightarrow C_{1n}.R_0^n+D_{1n}.R_0^{-n}=0\Rightarrow D_{1n}=-C_{1n}.(R_0)^{2n}$$

$$\Rightarrow R(r)=C_n[r^n-(R_0)^{2n}r^{-n}]=C_nR_0^n\left[\left(\frac{r}{R_0}\right)^n-\left(\frac{r}{R_0}\right)^{-n}\right]$$

$$\Rightarrow T_1(r,\varphi)=\sum_{n=0}^{\infty}\left(a_{1n}\sin(n\varphi)+b_{1n}\cos(n\varphi)\right)R_0^n\left[\left(\frac{r}{R_0}\right)^n-\left(\frac{r}{R_0}\right)^{-n}\right]$$

$$\Rightarrow T_1(R_i,\varphi)=T_i(\varphi)=\sum_{n=0}^{\infty}\left(a_{1n}\sin(n\varphi)+b_{1n}\cos(n\varphi)\right)R_0^n\left[\left(\frac{r}{R_0}\right)^n-\left(\frac{r}{R_0}\right)^{-n}\right]$$

$$\Rightarrow a_{1n}=\frac{\int_0^{2\pi}T_i(\varphi).sin(n\varphi)d\varphi}{\int_0^{2\pi}sin^2(n\varphi)d\varphi.R_0^n\left[\left(\frac{R_i}{R_0}\right)^n-\left(\frac{R_i}{R_0}\right)^{-n}\right]}$$

$$\Rightarrow b_{1n}=\frac{\int_0^{2\pi}T_i(\varphi).cos(n\varphi)d\varphi}{\int_0^{2\pi}cos^2(n\varphi)d\varphi.R_0^n\left[\left(\frac{R_i}{R_0}\right)^n-\left(\frac{R_i}{R_0}\right)^{-n}\right]}$$

برای $T_2(r,\varphi)$:

$$T_2(r,\varphi)=\sum_{n=0}^{\infty}\left(a_{2n}\sin(n\varphi)+b_{2n}\cos(n\varphi)\right)R_i^n\left[\left(\frac{r}{R_0}\right)^n-\left(\frac{r}{R_0}\right)^{-n}\right]$$

$$\Rightarrow T_2(R_0,\varphi)=T_o(\varphi)=\sum_{n=0}^{\infty}\left(a_{2n}\sin(n\varphi)+b_{2n}\cos(n\varphi)\right)R_i^n\left[\left(\frac{R_0}{R_i}\right)^n-\left(\frac{R_0}{R_i}\right)^{-n}\right]$$

$$a_{2n}=\frac{\int_0^{2\pi}T_0(\varphi).\sin(n\varphi)d\varphi}{\int_0^{2\pi}\cos^2(n\varphi)d\varphi.R_i^n\left[\left(\frac{R_0}{R_i}\right)^n-\left(\frac{R_0}{R_i}\right)^{-n}\right]}$$

و

$$b_{2n}=\frac{\int_0^{2\pi}T_0(\varphi).\sin(n\varphi)d\varphi}{\int_0^{2\pi}\cos^2(n\varphi)d\varphi.R_i^n\left[\left(\frac{R_0}{R_i}\right)^n-\left(\frac{R_0}{R_i}\right)^{-n}\right]}$$

مسئاله ۱۷-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{1}{r^2}\frac{d^2T}{d\varphi^2}=0\,,BC\begin{cases}T(R_i,\varphi)=T_\infty\\ \frac{dT}{dr}(R_0,\varphi)=\begin{cases}-\frac{q''}{k} & for\quad 0<\varphi<\pi\\ 0 & for\quad 0<\varphi<2\pi\end{cases}\\ T(r,\varphi)=T(r,\varphi+2\pi)\\ \frac{dT(r,\varphi)}{rd\varphi}=\frac{dT(r,\varphi+2\pi)}{rd\varphi}\end{cases}$$

$$\theta=T-T_\infty$$

$$\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{1}{r^2}\frac{d^2\theta}{d\varphi^2}=0\,,BC\begin{cases}\theta(R_i,\varphi)=0\\ \frac{d\theta}{dr}(R_0,\varphi)=\begin{cases}-\frac{q''}{k} & 0<\varphi<\pi\\ 0 & \pi<\varphi<2\pi\end{cases}\\ \theta(r,\varphi)=\theta(r,\varphi+2\pi)\\ \frac{d\theta(r,\varphi)}{rd\varphi}=\frac{d\theta(r,\varphi+2\pi)}{rd\varphi}\end{cases}$$

$$\theta(r,\varphi)=R(r).\phi(\varphi)$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{\varphi}\frac{d^2\phi}{d\varphi^2}=+\lambda_n^2$$

$$\phi''+\lambda_n^2\phi=0\Rightarrow\phi(\varphi)=A_n\sin\lambda_n\varphi+B_n\cos\lambda_n\varphi$$

$$\frac{d\phi(\varphi)}{r\varphi}=\frac{d\phi(\varphi+2\pi)}{d\varphi}\Rightarrow\lambda_n=n\,,n=0,1,2,\ldots$$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda_n^2R=0\xrightarrow{Euler}t^2+(a-1)t-\lambda_n^2=0\Rightarrow t^2-\lambda_n^2=0$$

$$\Rightarrow t=\pm\lambda_n\Rightarrow R(r)=C_nr^{\lambda_n}+D_nr^{-\lambda_n}\Rightarrow R(r)=C_nr^n+D_nr^{-n}$$

$$R(R_i)=0\Rightarrow C_nR_i^n+D_nR_i^{-n}=0\Rightarrow D_n=-C_n(R_i)^{2n}$$

$$\Rightarrow R(r)=C_n[r^n-R_i^{2n}r^{-n}]=C_nR_i^n\left[\left(\frac{r}{R_i}\right)^n-\left(\frac{r}{R_i}\right)^{-n}\right]$$

$$\theta(r,\varphi) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\varphi) + b_n\sin(n\varphi)\right)R_i^n\left[\left(\frac{r}{R_i}\right)^n - \left(\frac{r}{R_i}\right)^{-n}\right]$$

$$\underbrace{\frac{d\theta}{dr}(R_0,\varphi)}_{f(\varphi)} = \sum_{n=1}^{\infty}\left(a_n\cos(n\varphi) + b_n\sin(n\varphi)\right)\frac{R_i^n}{R_0}\left[\left(\frac{R_o}{R_i}\right)^n + \left(\frac{R_o}{R_i}\right)^{-n}\right]n$$

$$\Rightarrow a_n = \frac{\int_0^{2\pi} f(\varphi).\cos(n\varphi)d\varphi}{\int_0^{2\pi}\cos^2(n\varphi)d\varphi} = \frac{\int_0^{\pi}\frac{q''}{k}\cos\varphi\, d\varphi + \int_0^{2\pi} 0\, d\varphi}{\int_0^{2\pi}\cos^2(n\varphi)d\varphi}$$

$$\Rightarrow a_n = \frac{\frac{q''}{k}(\sin\pi - \sin 0)}{\int_0^{2\pi}\cos^2(n\varphi)d\varphi.\frac{R_i^n}{R_0}\left[\left(\frac{R_o}{R_i}\right)^n + \left(\frac{R_o}{R_i}\right)^{-n}\right]n}$$

$$b_n = \frac{\int_0^{2\pi} f(\varphi).\sin(n\varphi)d\varphi}{\int_0^{2\pi}\sin^2(n\varphi)d\varphi} = \frac{\int_0^{\pi}\frac{q''}{k}\sin(n\varphi)d\varphi}{\pi} = \frac{2q''}{nk\pi}$$

$$\Rightarrow b_n = \frac{2q''}{nk\pi\frac{R_i^n}{R_0}\left[\left(\frac{R_o}{R_i}\right)^n + \left(\frac{R_o}{R_i}\right)^{-n}\right]n}, n = even$$

$$\theta(r,\varphi) = T(r,\varphi) - T_\infty = \frac{2q''}{k\pi}\sum_{n=1}^{\infty}\frac{\left(\frac{r}{R_0}\right)^n - \left(\frac{r}{R_0}\right)^{-n}}{n^2\frac{r}{R_0}\left[\left(\frac{R_o}{R_i}\right)^n + \left(\frac{R_o}{R_i}\right)^{-n}\right]}\sin(n\varphi)$$

مساله ۱۸-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{1}{r^2}\frac{d^2T}{d\phi^2} = 0\ , BC\begin{cases}\frac{dT(0,\varphi)}{dr} = 0\ \ or\ \ T(0,\varphi) = finite\\ -k\frac{dT(R,\varphi)}{dr} = h(T - T_\infty)\ ,\\ \frac{dT(r,0)}{rd\varphi} = 0\\ -k\frac{dT(r,\varphi_0)}{rd\varphi} + q''(r) = 0\end{cases}$$

$$\theta = T - T_\infty$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{1}{r^2}\frac{d^2\theta}{d\varphi^2} = 0\ , BC\begin{cases}\frac{d\theta(0,\varphi)}{dr} = 0\\ -k\frac{d\theta(R,\varphi)}{dr} = h\theta\\ \frac{d\theta(r,0)}{rd\varphi} = 0\\ -k\frac{d\theta(r,\varphi_0)}{rd\varphi} = -q''(r)\end{cases}$$

$$\theta(r,\varphi) = R(r).\phi(\varphi)$$

$$\frac{d^2\phi}{d\varphi^2} - \lambda_n^2\phi = 0\ , \frac{d\phi(0)}{d\varphi} = 0$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda_n^2 R = 0\ \text{Euler}\ , BC\begin{cases}R(0) = finite\\ \frac{dR}{dr}(R) = -\frac{h}{k}R(R)\end{cases}$$

$$\phi(\varphi) = A_n\sinh(\lambda_n\varphi) + B_n\cosh(\lambda_n\varphi)$$

$$\frac{d\phi(0)}{d\varphi} = 0 = \lambda_n(A_n\times 1 + 0) = 0 \Rightarrow A_n = 0 \Rightarrow \phi(\varphi) = B_n\cos\lambda_n\varphi$$

و

$$R(r) = C_n\cos(\lambda_n\, lnr) + D_n\sin(\lambda_n\, lnr)$$

$$r = R \Rightarrow \frac{dR}{dr} = -\frac{h}{k}R \Rightarrow \frac{\lambda_n}{R}\left(-C_n\sin(\lambda_n\ln R) + D_n\cos(\lambda_n\ln R)\right) =$$

$$-\frac{h}{k}\left(C_n\cos(\lambda_n\ln R) + D_n\sin(\lambda_n\ln R)\right)$$

$$\Rightarrow C_n = -D_n\frac{\frac{h}{k}\sin(\lambda_n\, lnR) + \frac{\lambda_n}{k}\cos(\lambda_n\, lnR)}{\frac{h}{k}\cos(\lambda_n\, lnR) - \frac{\lambda_n}{k}\sin(\lambda_n\, lnR)} = -D_n\xi$$

$$R(r) = D_n\left[-\xi\cos(\lambda_n\ln R) + \sin(\lambda_n\ln R)\right]$$

$$\theta(r,\varphi) = \sum_{n=0}^{\infty} a_n\left[-\xi\cos(\lambda_n\ln R) + \sin(\lambda_n\ln R)\right]\cosh(\lambda_n\varphi)$$

$$k\frac{d\theta(r,\varphi_0)}{rd\varphi} = -q''(r) \Rightarrow \frac{d\theta(r,\varphi_0)}{d\varphi} = \sum_{n=0}^{\infty} a_n\left[-\xi\cos(\lambda_n\ln r) +\right.$$

$$\left.\sin(\lambda_n\ln R)\right]\lambda_n\sinh(\lambda_n\varphi_0)$$

$$k\frac{d\theta(r,\varphi_0)}{rd\varphi}=\sum_{n=0}^{\infty}a_n\frac{\left[-\xi\cos(\lambda_n \ln r)+\sin(\lambda_n \ln r)\right]}{r}\lambda_n\sinh(\lambda_n\varphi_0)=q''(r)$$

مساله ۱۹-۴)

(a

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{d^2T}{dx^2}=0\ ,BC\begin{cases}T(0,x)=finite\\ T(R,x)=T_\infty\\ T\left(r,\frac{L}{2}\right)=T_0\\ \frac{dT}{dx}(r,0)=0\end{cases}$$

$$\theta=T-T_\infty$$

$$\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{d^2\theta}{dx^2}=0\ ,BC\begin{cases}\left.\begin{matrix}\theta(0,x)=finite\\ \theta(R,x)=0\end{matrix}\right\}\text{همگن}\\ \left.\begin{matrix}\theta\left(r,\frac{L}{2}\right)=T_0-T_\infty\\ \frac{d\theta}{dx}(r,0)=0\end{matrix}\right\}\text{ناهمگن}\end{cases}$$

$$\theta(r,x)=R(r).X(x)$$

$$\frac{1}{r}\frac{d}{dr}\left(rX\frac{dR}{dr}\right)+R.\frac{d^2X}{dx^2}=0\Rightarrow\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda_n^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right)+rR\lambda_n^2=0\Rightarrow r\frac{d^2R}{dr^2}+\frac{dR}{dr}+rR\lambda_n^2=0\Rightarrow r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+$$
$$r^2\lambda_n^2R=0$$

تابع بسل: $R(r)=A_nJ_0(\lambda_n r)+B_nY_0(\lambda_n r)$

$\frac{dR(0)}{dr}=0$ یا $R(0)=finite\Rightarrow B_n=0\Rightarrow R(r)=A_nJ_0(\lambda_n r)$

$R(R)=0\Rightarrow R(R)=A_nJ_0(\lambda_n r)=0\Rightarrow$ λ_n حاصل خواهد شد

$$\frac{d^2X}{dx^2}-\lambda_n^2X=0\Rightarrow m^2-\lambda_n^2=0\Rightarrow m^2=\lambda_n^2\Rightarrow m=\pm\lambda_n$$

$$X(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$\frac{dX}{dx}(0)=0\Rightarrow C_n\lambda_n\times 1+D_n\times 0=0\Rightarrow C_n=0$$

$$X(x)=D_n\cosh(\lambda_n x)$$

$$\theta(r,x)=\sum_{n=0}^{\infty}a_n.J_0(\lambda_n r).\cosh(\lambda_n x)$$

$$\theta\left(r,\frac{L}{2}\right)=T_0-T_\infty=\sum_{m=0}^{\infty}a_n.J_0(\lambda_n r).\cosh\left(\lambda_n\frac{L}{2}\right)$$

$$\Rightarrow a_n=\frac{(T_0-T_\infty)\int_0^R rJ_0(\lambda_n r)dr}{\cosh\left(\lambda_n\frac{L}{2}\right)\int_0^R rJ_0^2(\lambda_n r)dr}$$

(b

$$\Psi=T-T_0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right)+\frac{d^2\Psi}{dx^2}=0\ ,BC\begin{cases}\left.\begin{matrix}\Psi(0,x)=finite\\ \Psi(R,x)=T_\infty-T_0\end{matrix}\right\}\text{ناهمگن}\\ \left.\begin{matrix}\Psi\left(r,\frac{L}{2}\right)=0\\ \frac{d\Psi}{dx}(r,0)=0\end{matrix}\right\}\text{همگن}\end{cases}$$

$$\Psi(r,x)=R(r).X(x)$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=+\lambda_n^2$$

$$\frac{d^2X}{dx^2}+\lambda_n^2X=0\Rightarrow m^2+\lambda_n^2=0\Rightarrow m=\pm i\lambda_n$$

$$X(x)=A_n\sin(\lambda_n x)+B_n\cos(\lambda_n x)$$

$$\frac{dX(0)}{dx}=0\Rightarrow A_n\lambda_n\cos(\lambda_n 0)-B_n\lambda_n\sin(\lambda_n 0)\Rightarrow A_n=0$$

$$\Rightarrow X(x)=B_n\cos(\lambda_n x)$$

$$X\left(\frac{L}{2}\right)=B_n\cos\left(\lambda_n\frac{L}{2}\right)=0\Rightarrow\lambda_n\frac{L}{2}=\frac{(n+1)\pi}{2}\Rightarrow\lambda_n=\frac{(n+1)\pi}{L}$$

$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-r^2\lambda_n^2R=0\Rightarrow$ تابع بسل

$$\Rightarrow R(r)=C_nI_0(\lambda_n r)+D_nk_0(\lambda_n r)$$

$$R(0)=finite\Rightarrow D_n=0\Rightarrow R(r)=C_nI_0(\lambda_n R)$$

$$\Rightarrow\Psi(r,x)=\sum_{m=0}^{\infty}b_n.\cos(\lambda_n x).I_0(\lambda_n r)$$

$$\Psi(R,x) = T_\infty - T_0 = \sum_{n=0}^{\infty} b_n.\cos(\lambda_n x).I_0(\lambda_n R)$$

$$\Rightarrow b_n = \frac{(T_\infty - T_0)\int_0^{\frac{L}{2}}\cos(\lambda_n x) I_0(\lambda_n R)dx}{\int_0^{\frac{L}{2}}\cos^2(\lambda_n x)dx}$$

مساله ۲۰-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{d^2\theta}{dx^2} + \frac{u'''}{k} = 0\ ,\ BC\begin{cases}\theta(0,x) = finite\\ -k\frac{d\theta}{dr}(R,0) = h\theta(R,x)\\ -k\frac{d\theta}{dx}\left(r,\frac{L}{2}\right) = h\theta\left(r,\frac{L}{2}\right)\\ \frac{d\theta}{dx}(r,0) = 0\end{cases},\ \theta = T - T_\infty$$

$$(I)\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{d^2\phi}{dx^2} = 0\ ,\ BC\begin{cases}\left.\begin{matrix}\phi(0,x) = finite\\ -k\frac{d\phi}{dr}(R,x) = h\phi(R,x)\end{matrix}\right\}\text{همگن}\\ \left.\begin{matrix}-k\frac{d\phi}{dr}\left(r,\frac{L}{2}\right) = h(\phi+\Psi)\\ \frac{d\phi}{dx}(r,0) = 0\end{matrix}\right\}\text{ناهمگن}\end{cases}$$

$$(II)\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) + \frac{u'''}{k} = 0\ ,\ BC\begin{cases}\frac{d\Psi}{dr}(0) = 0\ \ or\ \ \Psi(0) = finite\\ -k\frac{d\Psi}{dr}(R) = h\Psi(R)\end{cases}$$

$$(II) \Rightarrow r\frac{d\Psi}{dr} = -\frac{u'''r^2}{2k} + C_1 \Rightarrow \frac{d\Psi}{dr} = \frac{-u'''r}{2k} + C_1 Lnr$$

$$\Rightarrow \Psi(r) = \frac{u'''r^2}{4k} + C_1(-r + rLnr) + C_2$$

$$r = 0 \Rightarrow C_1 = 0\ ,\ r = R \Rightarrow -k\frac{d\Psi}{dr} = h\Psi \Rightarrow C_2 = \frac{u'''R^2}{4k}\left(1 + \frac{2k}{Rh}\right)$$

$$\Rightarrow \Psi(r) = \frac{u'''}{4k}\left(R^2 - r^2 + \frac{2Rk}{h}\right)$$

$$(I) \Rightarrow \phi(r,x) = R(r).X(x) \Rightarrow \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{-1}{X}\frac{d^2X}{dx^2} = -\lambda_n^2$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + r^2\lambda_n^2 R = 0 \Rightarrow R(r) = A_n J_0(\lambda_n r) + B_n Y_0(\lambda_n r)$$

$$R(0) = finite \Rightarrow B_n = 0 \Rightarrow R(r) = A_n J_0(\lambda_n r)$$

$R(R) \Rightarrow \frac{J_0(\lambda_n R)}{J_1(\lambda_n R)} = \frac{k}{h}\lambda_n\ ,\ n = 1, 2, \ldots \Rightarrow$ با حل این معادله حاصل خواهد شد λ_n

$$\frac{d^2X}{dx^2} - \lambda_n^2 X = 0 \Rightarrow X(x) = C_n\sinh(\lambda_n x) + D_n\cosh(\lambda_n x)$$

$$x = 0 \Rightarrow \frac{dX}{dx} = \lambda(C_n \times 1 + D_n \times 0) = C_n\lambda_n = 0 \Rightarrow C_n = 0$$

$$\Rightarrow X(x) = D_n\cosh(\lambda_n x)$$

$$x = \frac{L}{2} \Rightarrow -k\frac{d\phi}{dx} = h(\phi - \Psi)$$

$$\phi(r,x) = \sum_{n=0}^{\infty} a_n.J_0(\lambda_n r).\cosh(\lambda_n x)$$

$$\Rightarrow -k\sum_{n=0}^{\infty} a_n.J_0(\lambda_n r).\lambda_n\sinh\left(\lambda_n\frac{L}{2}\right) =$$

$$h\left[\sum_{n=0}^{\infty} a_n.J_0(\lambda_n r)\cosh\left(\lambda_n\frac{L}{2}\right) - \frac{u'''R^2}{4k}\left(1 + \frac{2k}{Rh} - \left(\frac{r}{R}\right)^2\right)\right]$$

$$\frac{u'''R^2}{4k}\left(1 + \frac{2k}{Rh} - \left(\frac{r}{R}\right)^2\right) =$$

$$\sum_{n=0}^{\infty} a_n.J_0(\lambda_n r)\left[\frac{k}{h}\sinh\left(\lambda_n\frac{L}{2}\right) + \lambda_n\cosh\left(\lambda_n\frac{L}{2}\right)\right]$$

$$\int_0^R \frac{u'''R^2}{4k}\left(1 + \frac{2k}{Rh} - \left(\frac{r}{R}\right)^2\right) rJ_0(\lambda_n r)dr =$$

$$\underbrace{\frac{k}{h}\sinh\left(\lambda_n\frac{L}{2}\right) + \lambda_n\cosh\left(\lambda_n\frac{L}{2}\right)a_n}_{b_n}.\int_0^R rJ_0^2(\lambda_n r)dr$$

$$b_n = \frac{\frac{u'''R^2}{4k}\left(\int_0^R rJ_0(\lambda_n r).\left(\frac{2k}{Rh}+1\right)dr + \int_0^R r^3 J_0(\lambda_n r)dr\left(-\frac{1}{R^2}\right)\right)}{\int_0^R rJ_0^2(\lambda_n r)dr}$$

$$\theta(r,x) = \phi(r,x) + \Psi(r) =$$

$$\sum_{n=0}^{\infty}\frac{b_n}{\frac{k}{h}\sinh\left(\lambda_n\frac{L}{2}\right) + \lambda_n\cosh\left(\lambda_n\frac{L}{2}\right)}.J_0(\lambda_n r).\cosh(\lambda_n x) + \frac{u'''}{4k}\left(R^2 - r^2 + \frac{2Rk}{h}\right)$$

مسئله ۲۱-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{d^2\theta}{dx^2}=0\,,\ BC\begin{cases}\left.\begin{aligned}&\theta(0,x)=finite\\&\frac{d\theta}{dr}(R,x)=0\end{aligned}\right\}\text{همگن}\\[2ex]\left.\begin{aligned}&\theta\left(r,\frac{H}{2}\right)=0\\&\theta\left(r,-\frac{H}{2}\right)=\theta_0\end{aligned}\right\}\text{ناهمگن}\end{cases}$$

$$\theta(r,x)=R(r).X(x)$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda_n^2$$

$$R(r)=A_nJ_0(\lambda_n r)+B_nY_0(\lambda_n r)$$

$$R(0)=finite\Rightarrow B_n=0\Rightarrow R(r)=A_nJ_0(\lambda_n r)$$

$\frac{dR}{dr}(R)=0\Rightarrow A_nJ_0'(\lambda_n R)=0\Rightarrow$ با حل این معادله حاصل خواهد شد λ_n

$$X(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$X\left(\frac{H}{2}\right)=0\Rightarrow C_n\sinh\left(\lambda_n\frac{H}{2}\right)=-D_n\cosh\left(\lambda_n\frac{H}{2}\right)$$

$$\Rightarrow D_n=-C_n\tanh\left(\lambda_n\frac{H}{2}\right)$$

$$\Rightarrow X(x)=C_n\left(\sinh(\lambda_n x)-\tanh\left(\lambda_n\frac{H}{2}\right).\cosh(\lambda_n x)\right)$$

$$\theta(r,x)=\sum_{n=0}^{\infty}a_n.\left(\sinh(\lambda_n x)-\tanh\left(\lambda_n\frac{H}{2}\right).\cosh(\lambda_n x)\right).J_0(\lambda_n r)$$

$$\theta\left(r,-\frac{H}{2}\right)=\theta_0=$$

$$\sum_{n=0}^{\infty}\underbrace{a_n\left(-\sinh\left(\lambda_n\frac{H}{2}\right)-\tanh\left(\lambda_n\frac{H}{2}\right).\cosh\left(\lambda_n\frac{H}{2}\right)\right)}_{b_n}.J_0(\lambda_n r)$$

$$\Rightarrow b_n=\frac{\theta_0\int_0^R rJ_0(\lambda_n r)dr}{\int_0^R r_0^2(\lambda_n r)dr}$$

مسئله ۲۲-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{d^2T}{dx^2}=0\,,\ BC\begin{cases}T(0,x)=finite\\-k\frac{dT}{dr}(R,x)=h[T(R,Z)-T_\infty]\\T(r,\infty)=T_\infty\\-k\frac{dT}{dr}(r,x)=\begin{cases}0 & 0\le r\le R_0\\ \mu p\bar{R}\omega & R_0<r<R\end{cases}\end{cases}$$

$$q=\int_{R_0}^{R}\frac{\mu p(r\omega)(2\pi r dr)}{\pi(R^2-R_0^2)}=\frac{2}{3}\mu p\omega\frac{R^3-R_0^3}{R^2-R_0^2}$$

$$\theta=T-T_\infty\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{d^2\theta}{dx^2}=0$$

$$BC\begin{cases}\theta(0,x)=finite\\-k\frac{d\theta}{dr}(R,x)=h\theta(R,Z)\\\theta(r,\infty)=0\\-k\frac{d\theta}{dr}(r,x)=\begin{cases}0 & 0\le r\le R_0\\ \mu p\bar{R}\omega & R_0<r<R\end{cases}\end{cases}$$

$$\theta(r,x)=R(r).X(x)$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda_n^2$$

$$\Rightarrow R(r)=A_nJ_0(\lambda_n r)+B_nY_0(\lambda_n r)$$

$$R(0)=finite\Rightarrow B_n=0\Rightarrow R(r)=A_nJ_0(\lambda_n r)$$

$$-k\frac{dR}{dr}(R)=hR(R)\Rightarrow J_1(\lambda_n R)=\frac{hR}{k\lambda_n R}\left(J_0(\lambda_n r)\right)=\frac{Bi}{\lambda_n R}J_0(\lambda_n R)$$

$$\Rightarrow X(x)=C_ne^{-\lambda_n x}+D_ne^{\lambda_n x}$$

$$X(\infty)=0\Rightarrow D_n=0\Rightarrow X(x)=C_ne^{-\lambda_n x}$$

$$\Rightarrow \theta(r,x) = \sum_{n=0}^{\infty} a_n . e^{-\lambda_n x} . J_0(\lambda_n r)$$

$$0 \le r \le R_0 \Rightarrow 0 = \sum_{n=0}^{\infty} a_n . J_0(\lambda_n r)$$

$$R_0 < r < R \Rightarrow q = \underbrace{\frac{2}{3}\mu p\omega \frac{R^3 - R_0^3}{R^2 - R_0^2}}_{f} = \sum_{n=0}^{\infty} a_n . J_0(\lambda_n r)$$

$$a_n = \frac{\int_{R_0}^{R} f . r J_0(\lambda_n r) dr}{\int_{R_0}^{R} r y_0^2(\lambda_n r) dr}$$

مساله ۲۳-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{d^2T}{dx^2} = 0 \,,\, BC \begin{cases} T(r,0) = f(r) \\ T(r,\infty) = T_\infty \\ T(0,x) = finite \\ T(R,x) = 0 \end{cases}$$

$$\theta = T - T_\infty \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{d^2\theta}{dx^2} = 0 \,,\, BC \begin{cases} \theta(r,0) = f(r) \\ \theta(r,\infty) = 0 \\ \theta(0,x) = finite \\ \theta(R,x) = 0 \end{cases}$$

تقریب درجه اول: $\theta(r,x) = (a_2 r^2 + a_1 r + a_0)$

$$r = 0 \Rightarrow \theta(0,x) = finite \Rightarrow \frac{d\theta}{dr} = 0 \Rightarrow a_1 = 0$$

$$r = R \Rightarrow \theta(R,x) = 0 \Rightarrow a_2 R^2 + a_0 = 0 \Rightarrow a_0 = -a_2 r^2$$

$$\Rightarrow \theta(r,x) = \underbrace{a_2}_{f(x)} (r^2 - R^2)$$

$$\int_0^R \frac{d}{dr}\left(r\frac{d\theta}{dr}\right) dr + \int_0^R r\frac{d^2\theta}{dx^2} dr = 0 \Rightarrow r\frac{d\theta}{dr}\Big|_R - r\frac{d\theta}{dr}\Big|_0 + \frac{d^2 a_2}{dx^2}\int_0^R r(r^2 - R^2)dr = 0$$

$$2a_2 R^2 - 0 + \frac{d^2 a_2}{dx^2}\left(\frac{R^2}{4} - \frac{R^4}{2}\right) = 0 \Rightarrow \frac{d^2 a_2}{dx^2} - 4\frac{a^2}{R^2} = 0$$

$$\Rightarrow a_2 = C_1 e^{-\frac{2}{R}x} + C_2 e^{\frac{2}{R}x}$$

$$\theta(r,\infty) = 0 \Rightarrow C_2 = 0 \Rightarrow a_2(x) = C_1 e^{-\frac{2}{R}x}$$

$$\Rightarrow \theta(r,x) = Ce^{-\frac{2}{R}x}(r^2 - R^2)$$

$$\theta(r,0) = f(r) \Rightarrow C(r^2 - R^2) = f(r) \Rightarrow C = \frac{f(r)}{r^2 - R^2}$$

$$\Rightarrow \theta(r,x) = \frac{f(r)}{r^2 - R^2} e^{-\frac{2}{R}x}(r^2 - R^2) = f(r). e^{-\frac{2}{R}x}$$

تقریب درجه دوم: $\theta(r,z) = (r^2 - R^2)(a + br^2)$

$$a = f(x) \,,\, b = g(x)$$

$$\int_0^R \frac{d}{dr}\left(r\frac{d\theta}{dr}\right) dr + \int_0^R r\frac{d^2\theta}{dx^2} dr = 0 \Rightarrow r\frac{d\theta}{dr}\Big|_R - r\frac{d\theta}{dr}\Big|_0 + \frac{d^2 a_2}{dx^2}\int_0^R r(r^2 - R^2)dr + \frac{d^2 b_2}{dx^2}\int_0^R r(r^4 - r^2R^2)dr = 0$$

$$\Rightarrow 2aR^2 + 4bR^4 - 2bR^4 + \left(\frac{R^4}{4} - \frac{R^2}{2}\right)a'' + \left(\frac{R^6}{6} - \frac{R^6}{4}\right)b'' = 0$$

با استفاده از اپراتور D:

$$\left(-\frac{R^4}{4}D^2 + 2R^2\right)a + \left(-\frac{R^6}{12}D^2 + 2D^4\right)b = 0$$

$$(I) \Rightarrow \left(D^2 - \frac{8}{R^2}\right)a + \left(-\frac{R^3}{3}D^2 - 8\right)b = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) = \frac{1}{r}\frac{d}{dr}\left(2ar^2 + b(4r^4 - 2r^2R^2)\right) = \frac{1}{r}\left(4ar + b(8r^3 - 4rR)\right) = 4a + 8br^2 - 4bR$$

$$\frac{d^2\theta}{dx^2} = a''(r^2 + R^2) + b''r^2(r^2 - R^2)$$

$$\text{for } r = 0 \Rightarrow 4a + 4bR - R^2 a'' = 0$$

$$(II) \Rightarrow \left(D^2 - \frac{4}{R^2}\right)a - \frac{4}{R}b = 0$$

$$(I),(II) \Rightarrow \left[\frac{R^3}{12}D^4 + \left(1 - \frac{2}{R} - \frac{R}{3}\right)D^2 + \frac{8}{R} - \frac{2}{R^2}\right]a = 0 \,,\, \alpha < 0 < B$$

$$a = C_1 e^{\alpha x} + C_2 e^{\beta x} \,,\, x \to \infty \Rightarrow \theta(r,\infty) = 0 \Rightarrow C_1 = 0$$

$$b = \frac{R}{4}\left(C_2\beta^2 e^{\beta x} - \frac{4}{R^2}e^{\beta x}\right)$$

$\theta(r,0) = f(r) \Rightarrow \frac{R}{4}\left(C_2\beta^2 - \frac{4}{R^2}\right) = f(r) \Rightarrow$ به دست خواهد آمد C_2

$$T(r,z) = C_2(r^2 - R^2)\left(1 + \left(\beta^2 - \frac{4}{R^2}\right)r^2\right)e^{\beta x}$$

مساله ۴-۲۴

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{d^2T}{dx^2} = 0 \,,\; BC\begin{cases} \frac{dT(0,x)}{dr} = 0 \\ \mu pR\omega = h_1(T(R,x) - T_\infty) + k\frac{dT(R,x)}{dr} \\ \frac{dT(r,0)}{dx} = 0 \\ -k\frac{dT\left(r,\frac{L}{2}\right)}{dx} = h_2\left(T\left(r,\frac{L}{2}\right) - T_\infty\right) \end{cases}$$

$$\theta = T - T_\infty \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{d^2\theta}{dx^2} = 0$$

$$BC\begin{cases} \left.\begin{array}{l} \frac{d\theta(0,x)}{dr} = 0 \\ \mu pR\omega = h_1\theta(R,x) + k\frac{d\theta(R,x)}{dr} \end{array}\right\} \text{ناهمگن} \\ \left.\begin{array}{l} \frac{d\theta(r,0)}{dx} = 0 \\ -k\frac{d\theta\left(r,\frac{L}{2}\right)}{dx} = h_2\theta\left(r,\frac{L}{2}\right) \end{array}\right\} \text{همگن} \end{cases}$$

$$\theta(r,x) = R(r).X(x)$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{X}\frac{d^2X}{dx^2} = 0 \Rightarrow \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{X}\frac{d^2X}{dx^2} = +\lambda_n^2$$

$$\Rightarrow X'' + \lambda_n^2 X = 0 \Rightarrow X(x) = A_n\sin(\lambda_n x) + B_n\cos(\lambda_n x)$$

$$r^2R'' + rR' - \lambda_n^2 r^2 R = 0 \Rightarrow R(r) = C_n I_0(\lambda_n r) + D_n k_0(\lambda_n r)$$

$$\frac{dT(0,x)}{dr} = 0 \Rightarrow \frac{dR(0)}{dr} = 0 \Rightarrow D_n = 0 \Rightarrow R(r) = C_n I_0(\lambda_n r)$$

$$\frac{dX}{dx}(0) = 0 \Rightarrow A_n\lambda_n\cos(0) - B_n\lambda_n\sin(0) = 0 \Rightarrow A_n = 0$$

$$\Rightarrow X(x) = B_n\cos(\lambda_n x)\,,\; -kX'\left(\frac{L}{2}\right) = h_2X\left(\frac{L}{2}\right) \Rightarrow \frac{k}{h_2}\lambda_n = \cot\left(\frac{\lambda_n L}{2}\right)$$

$$\theta(r,x) = \sum_{n=0}^{\infty} a_n.\cos(\lambda_n x).I_0(\lambda_n r)$$

$$\mu p(R\omega) = h_1\theta(R,x) + k\frac{d\theta(R,x)}{dr}$$

$$\mu p(R\omega) =$$

$$h_1\sum_{n=0}^{\infty} a_n.\cos(\lambda_n x).I_0(\lambda_n R) + k\sum_{n=0}^{\infty} a_n.\cos(\lambda_n x)\lambda_n I_1(\lambda_n R)$$

$$\mu p(R\omega) = \sum_{n=0}^{\infty} a_n.\cos(\lambda_n x)\left[h_1I_0(\lambda_n R) + k\lambda_n I_1(\lambda_n R)\right]$$

$$\Rightarrow a_n = \frac{\mu p(R\omega)}{[h_1I_0(\lambda_n R) + k\lambda_n I_1(\lambda_n R)]}.\frac{\int_0^{\frac{L}{2}}\cos(\lambda_n R)dx}{\int_0^{\frac{L}{2}}\cos^2(\lambda_n x)dx}$$

مساله ۴-۲۵

برای میله جامد: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dT}\right) + \frac{u'''}{k_2} = 0$

برای سردساز: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right) = \frac{pvc}{k_1} = \frac{dT_1}{dz}$

$$BC\begin{cases} (1)\;\; T_2(0) = finite \\ (2)\;\; T_1(R,z) = T_2(R) \\ (3)\;\; T_1(r,0) = T_0 \\ (4)\;\; \frac{dT_1}{dr}(R',z) = 0 \\ (5)\;\; -k_2\frac{dT_2(R)}{dr} = -k_2\frac{dT_1}{dr}(R,z) \end{cases}$$

حل مساله به عهده خواننده گذاشته می‌شود.

مساله ۴-۲۶

برای $z > 0$: $\frac{d^2T_1}{dz^2} + \frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right) - \frac{\rho vc}{k}\frac{dT_1}{dz} = 0$

برای $z < 0: \frac{d^2T_2}{dz^2} + \frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right) - \frac{\rho vc}{k}\frac{dT_2}{dz} = 0$

$$\frac{\rho vc}{k} = 2s\,, BC\begin{cases}(1)\ \frac{dT_2(Z,R)}{dr} = 0\\ (2)\ \frac{dT_2(Z,0)}{dr} = 0\\ (3)\ T_2(-\infty, r) = 0\\ (4)\ \frac{dT_2(0,r)}{dz} = \frac{dT_1(0,r)}{dz}\\ (5)\ T_2(0,r) = T_1(0,r)\\ (6)\ \frac{q''}{k} = \frac{dT_1(Z,R)}{dr}\\ (7)\ \frac{dT_1(Z,0)}{dr} = 0\\ (8)\ T_1(\infty, r) \propto Z\end{cases}$$

for $z < 0: T_2(z,r) = Z_2(z).R_2(r)$

$\Rightarrow \frac{z''_2 - 2sz'_2}{z_2} = -\frac{R''_2}{R'_2} = +\lambda_n^2 \Rightarrow R_2(r) = A_{2n}.\sin(\lambda_n r) + B_{2n}.\cos(\lambda_n r)$

$(2): \frac{dR_2(0)}{dr} = 0 \Rightarrow A_{2n} = 0 \Rightarrow R_2(r) = B_{2n}.\cos(\lambda_n r)$

$(1): \frac{dR_2(R)}{dr} = 0 \Rightarrow \sin(\lambda_n r) = 0 \Rightarrow \lambda_n = \frac{n\pi}{R}, n = 1, 2, 3, \ldots$

$z''_2 - 2sz'_2 - \lambda_n^2 z_2 = 0$

$\Rightarrow z_2(z) = C_{2n}.e^{\left(s+\sqrt{s^2+\lambda_n^2}\right)z} + D_{2n}.e^{-\left(s+\sqrt{s^2+\lambda_n^2}\right)z}$

$(3): Z(-\infty) = 0 \Rightarrow D_{2n} = 0$

$\Rightarrow T_2(z,r) = \sum_{n=0}^{\infty} a_n.e^{\left(s+\sqrt{s^2+\lambda_n^2}\right)z}.\cos(\lambda_n r)$

for $z > 0\ : T_1(z,r) = \theta(z,r) = \phi(z,r) + p(z) + q(r)$

$$\begin{cases}\frac{d^2\phi}{dz^2} + \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - 2s\frac{d\phi}{dz} = 0 \quad (*)\\ \frac{d^2p}{dz^2} - 2s\frac{dp}{dz} = -\frac{1}{r}\frac{d}{dr}\left(r\frac{dq}{dr}\right) \quad (**)\end{cases}$$

$(**) \Rightarrow -\frac{1}{r}\frac{d}{dr}\left(r\frac{dq}{dr}\right) = C_1 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dq}{dr}\right) = -C_1\frac{d}{dr}\left(r\frac{dq}{dr}\right) = -C_1 r$

$r\frac{dq}{dr} = -C_1\frac{r^2}{2} + C_2 \Rightarrow \frac{dq}{dr} = -C_1\frac{r}{2} + \frac{c_2}{r} \Rightarrow q(r) = -C_1\frac{r^2}{4} + C_2 lnr + C_3$

$\frac{d^2p}{dz^2} - 2s\frac{dp}{dz} = C_1 \Rightarrow p(z) = A_1 + A_2e^{2sz} - \frac{c_1 z}{2s}$

به صورت اختیاری $A_1 = 0 \Rightarrow p(z) = A_2e^{2sz} - \frac{c_1 z}{2s}$

B.C 8: $T_1(\infty, r) \propto z \Rightarrow \phi(\infty, r) + p(\infty) + q(r) \propto z$

$\Rightarrow p(\infty) \propto z \Rightarrow A_2 = 0 \Rightarrow p(z) = \frac{c_1 z}{2s}$

B.C 7: $\frac{dT_1(z,0)}{dr} = 0 \Rightarrow \frac{d\phi(z,0)}{dr} + \frac{dq(0)}{dr} = 0\,, BC\begin{cases}\frac{d\phi(z,0)}{dr} = 0\\ \frac{dq(0)}{dr} = 0\end{cases}$

B.C 6: $\frac{q''}{k} = \frac{dT_1(z,R)}{dr} \Rightarrow \frac{q''}{k} = \frac{d\phi(z,R)}{dr} + \frac{dq(R)}{dr}\,, BC\begin{cases}\frac{d\phi(z,R)}{dr} = 0\\ \frac{dq(R)}{dr} = \frac{q''}{k}\end{cases}$

$q(r) = -C_1\frac{r^2}{4} + C_2 lnr + C_3$ فرض: $C_3 = 0$

$\frac{dq(0)}{dr} = 0 \Rightarrow \frac{dq}{dr} = -2C_1\frac{r}{4} + \frac{C_2}{r} = 0 \Rightarrow C_2 = 0$

$\frac{dq(R)}{dr} = \frac{q''}{k} \Rightarrow -C_1\frac{R^2}{4} = \frac{q''}{k} \Rightarrow C_1 = -\frac{4q''}{kR^2} \Rightarrow q(r) = \frac{4q''}{kR^2}.\frac{r^2}{4} = \frac{q''}{k}\left(\frac{r}{R}\right)^2$

$\frac{d^2\phi}{dz^2} + \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - 2s\frac{d\phi}{dz} = 0\,, BC\begin{cases}\frac{d\phi}{dr}(z,0) = 0\\ \frac{d\phi}{dr}(z,R) = 0\end{cases}$

$\phi(z,r) = z(z).R(r)$

$$\frac{Z''-2sZ'}{Z} = -\frac{R''}{R} = +\lambda_n^2 \Rightarrow \begin{cases} R'' + \lambda_n^2 R = 0 \\ Z'' - 2sZ' - \lambda_n^2 Z = 0 \end{cases}$$

$$\Rightarrow R(r) = A_n \sin(\lambda_n r) + B_n \cos(\lambda_n r)$$

$$Z(z) = C_n e^{\left(S+\sqrt{S^2+\lambda_n^2}\right)z} + D_n e^{-\left(S+\sqrt{S^2+\lambda_n^2}\right)z}$$

$$\frac{dR}{dr}(0) = 0 \Rightarrow A_n = 0,\ \frac{dR}{dr}(R) = 0 \Rightarrow \lambda_n = \frac{n\pi}{R},\ n = 0, 1, 2, \ldots$$

$$T_1(\infty, r) \propto z \Rightarrow \phi(\infty, r) \propto z \Rightarrow C_n = 0$$

$$\Rightarrow T_1(z,r) = \sum_{n=0}^{\infty} b_n . e^{\left(S-\sqrt{S^2+\lambda_n^2}\right)z} . \cos\lambda_n r + \frac{q'''}{k}\left(\frac{r}{R}\right)^2 + \frac{4q'''}{2skR^2} z$$

BC 5: $T_2(0,r) = T_1(0,r)$

$$\Rightarrow \sum_{n=0}^{\infty} a_n . \cos(\lambda_n r) = \sum_{n=0}^{\infty} b_n . \cos(\lambda_n r) + \frac{q'''}{kR^2} r^2$$

$$\Rightarrow \sum_{n=0}^{\infty} (a_n - b_n) . \cos(\lambda_n r) = \frac{q'''}{kR^2} r^2 \Rightarrow a_n - b_n = \frac{\frac{q'''}{kR^2} \int_0^R r^2 \cos(\lambda_n r) dr}{\int_0^R \cos^2(\lambda_n r) dr}$$

$$\int_0^R r^2 \cos(\lambda_n r) dr = \left(\frac{r^2}{\lambda_n} \sin(\lambda_n r) - \frac{2}{\lambda_n^3} \sin(\lambda_n r) + \frac{2r}{\lambda_n^2} \cos(\lambda_n r)\right)_0^R =$$

$$\underbrace{\frac{R^2}{\lambda_n} \sin(\lambda_n R)}_{0} - \underbrace{\frac{2}{\lambda_n^3} \sin(\lambda_n R)}_{0} + \frac{2R}{\lambda_n^2} \cos(\lambda_n R) = \frac{2R}{\lambda_n^2} \cos(\lambda_n R) = \frac{2R(-1)^n}{\lambda_n^2}$$

$$\sin(\lambda_n R) = 0 \Rightarrow \lambda_n = \frac{n\pi}{R}$$

$$\int_0^R \cos^2(\lambda_n r) dr = \left(\frac{r}{2} + \frac{\sin(2\lambda_n r)}{4\lambda_n}\right)_0^R = \frac{R}{2} - \frac{\overbrace{\sin(2\lambda_n R)}^{0}}{4\lambda_n} = \frac{R}{2}$$

$$a_n - b_n = \frac{\frac{q'''}{kR^2} \cdot \frac{2R(-1)^n}{\lambda_n^2}}{\frac{R}{2}} = \frac{4q'''(-1)^n}{kR^2 \lambda_n^2}$$

BC.4: $\frac{dT_2(0,r)}{dz} = \frac{dT_1(0,r)}{dz}$

$$\sum_{n=0}^{\infty} a_n . \left(s + \sqrt{s^2 + \lambda_n^2}\right) . \cos(\lambda_n r) =$$

$$\sum_{n=0}^{\infty} b_n . \left(s - \sqrt{s^2 + \lambda_n^2}\right) . \cos(\lambda_n r) + \frac{4q'''}{kskR^2}$$

$$\sum_{n=0}^{\infty} \left[\left(s + \sqrt{s^2 + \lambda_n^2}\right) a_n - \left(s - \sqrt{s^2 + \lambda_n^2}\right) b_n\right] . \cos(\lambda_n r) = \frac{2q'''}{skR^2}$$

$$\left(s + \sqrt{s^2 + \lambda_n^2}\right) a_n - \left(s - \sqrt{s^2 + \lambda_n^2}\right) b_n = \frac{\frac{2q'''}{skR^2} \int_0^R \cos(\lambda_n r) dr}{\int_0^R \cos^2(\lambda_n r) dr}$$

$$\int_0^R r^2 \cos(\lambda_n r) dr = \frac{1}{\lambda_n}\left(\underbrace{\sin\lambda_n R}_{0} - \sin\lambda_n 0\right) = 0$$

$$\Rightarrow a_n = \frac{\left(S+\sqrt{S^2+\lambda_n^2}\right)}{\left(S-\sqrt{S^2+\lambda_n^2}\right)} b_n\ ,\ a_n - b_n = \frac{4q'''(-1)^n}{kR^2 \lambda_n^2}$$

$$\left(\frac{S-\sqrt{S^2+\lambda_n^2}}{S+\sqrt{S^2+\lambda_n^2}} - 1\right) b_n = \frac{4q'''(-1)^n}{kR^2 \lambda_n^2} \Rightarrow b_n = \frac{-2q'''(-1)^n \left(S+\sqrt{S^2+\lambda_n^2}\right)}{R^2 k \lambda_n^2 \sqrt{S^2+\lambda_n^2}}$$

$$a_n = \frac{-2q'''(-1)^n \left(S-\sqrt{S^2+\lambda_n^2}\right)}{R^2 k \lambda_n^2 \sqrt{S^2+\lambda_n^2}}$$

$$\Rightarrow T_2(z,r) = \sum_{n=0}^{\infty} \frac{-2q'''(-1)^n \left(S-\sqrt{\lambda_n^2+S^2}\right)}{R^2 k \lambda_n^2 \sqrt{\lambda_n^2+S^2}} . e^{\left(S+\sqrt{S^2+\lambda_n^2}\right)z} . \cos(\lambda_n r)$$

$$\lambda_n = \frac{n\pi}{R},\ S = \frac{\rho vc}{2k},\ \left(S + \sqrt{S^2 + \lambda_n^2}\right) z = \frac{\rho vc}{2k} + \sqrt{\left(\frac{\rho vc}{2k}\right)^2 + \left(\frac{n\pi}{R}\right)^2}\, z$$

$$= \left(\frac{\rho vcR}{2k} + \sqrt{\left(\frac{\rho vc}{2k}\right)^2 + (n\pi)^2}\right)\left(\frac{z}{R}\right) = \left(\frac{vR}{2a} + \sqrt{\left(\frac{vR}{2a}\right)^2 + (n\pi)^2}\right)\left(\frac{z}{R}\right)$$

$$\frac{VR}{a} = P\ ,\ \frac{1}{\xi} = \frac{P}{\frac{z}{R}}\ ,\ \eta = \frac{r}{R} \Rightarrow \cos(\lambda_n r) = \cos(n\pi\eta)$$

$$\Rightarrow \left(S+\sqrt{S^2+\lambda_n^2}\right)z=\left(\frac{P}{2}+\sqrt{\left(\frac{P}{2}\right)^2+(n\pi)^2}\right)(P\eta)=\frac{1}{2}\left(1+\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}\right)P^2\eta$$

$$\Rightarrow T_2(z,r)=$$

$$\sum_{n=0}^{\infty}\frac{-2q'''(-1)^n\left(\frac{P}{2}\right)\left(1-\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}\right)P}{\frac{(n\pi)^2Pk}{2R^4}\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}}\exp\left(\frac{1}{2}\left(1+\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}P^2\eta\right)\right).\cos(n\pi\eta)$$

و

$$T_1(z,r)=$$

$$\sum_{n=0}^{\infty}\frac{-2q'''(-1)^n\left(\frac{P}{2}\right)\left(1-\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}\right)P}{\frac{(n\pi)^2Pk}{2R^4}\sqrt{1+\left(\frac{2n\pi}{P}\right)^2}}\exp\left(\frac{1}{2}\left(1-\sqrt{1-\left(\frac{2n\pi}{P}\right)^2}\right)P^2y\right).\cos(n\pi\eta)+\frac{q'''R^2}{k}\eta^2+\frac{\xi}{P}$$

مساله ۲۷-۴)

برای $z>0:\ \frac{\rho vc}{k}\frac{dT_2}{dz}=\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)$

برای $z<0:\ \frac{\rho vc}{k}\frac{dT_1}{dz}=\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)$

$$BC\begin{cases}(1)\quad T_1(r,-\infty)=T_0\\(2)\quad \frac{dT_1(0,z)}{dr}=0\\(3)\quad \frac{dT_1(0,z)}{dr}=0\\(4)\quad T_1(r,0)=T_2(r,0)\\(5)\quad \frac{dT_2(0,z)}{dr}=0\\(6)\quad \frac{u'''(R_0^2-R^2)}{2R}=k\frac{dT_2}{dr}(R,z)\end{cases}$$

برای $z<0$:

$$T_1(r,z)=R_1(r).Z_1(z)$$

$$\frac{pvc}{k}R_1\frac{dz_1}{dz}=\frac{1}{r}\frac{d}{dr}\left(rz_1\frac{dR_1}{dr}\right)$$

$$\frac{1}{\alpha z_1}\frac{dz_1}{dz}=\frac{1}{rR_1}\frac{d}{dr}\left(r\frac{dR_1}{dr}\right)=-\lambda_n^2$$

$$z_1(z)=C_{1n}e^{(-\alpha\lambda_n^2)z}\quad,\quad R(r)=A_{1n}J_0(\lambda_n r)+B_{1n}J_0(\lambda_n r)$$

$$(2):\frac{dT_1(0,z)}{dr}=0\Rightarrow B_{1n}=0\Rightarrow R(r)=A_{1n}J_0(\lambda_n r)$$

$$(3):\frac{dT_1(R,z)}{dr}=0\Rightarrow -A_{1n}\lambda_n J_1(\lambda_n R)=0\Rightarrow J_1(\lambda_n R)=0\Rightarrow \lambda_n$$

$$\Rightarrow T_1(r,z)=\sum_{n=0}^{\infty}a_n.J_0(\lambda_n R).e^{(-\alpha\lambda_n^2)z}$$

برای $z>0$: $T_2(r,z)=\phi(r,z)+q(r)$ حل این مساله مشابه مسائل پیشین است

مساله ۲۸-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{d^2T}{dz^2}=0\,,\,BC\begin{cases}\dfrac{dT}{dr}(0,z)=0\\ k\dfrac{dT}{dr}(R,z)=\begin{cases}\mu pR\omega & 0<z<l\\ 0 & l\le z\le L\end{cases}\\ \dfrac{dT}{dz}(r,0)=0\\ \dfrac{dT}{dz}(r,l)=0\end{cases}$$

$$T(r,z)=R(r).Z(z)$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{z}\frac{d^2T}{dz^2}=0$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{z}\frac{d^2z}{dz^2}=+\lambda_n^2$$

$$R(r)=A_nI_0(\lambda_n r)+B_nk_0(\lambda_n r)$$

$$\frac{dR}{dr}(0)=0\Rightarrow B_n=0\Rightarrow R(r)=A_nI_0(\lambda_n r)$$

$$Z(z)=C_n\sin(\lambda_n z)+D_n\cos(\lambda_n z)$$

$$\frac{dz}{dz}(0)=0\Rightarrow C_n=0\Rightarrow Z(z)=D_n\cos(\lambda_n z)$$

$$\frac{dz}{dz}(L)=0\Rightarrow -D_n\lambda_n\sin(\lambda_n L)=0\Rightarrow \lambda_n=\frac{n\pi}{L}\,,\,n=0,1\,,2,\ldots$$

$$T(r,z)=\sum_{n=0}^{\infty}a_n.\cos(\lambda_n z)I_0(\lambda_n z)$$

$$\text{for }\ 0<z<l\Rightarrow\frac{\mu pR\omega}{k}=\sum_{n=0}^{\infty}\lambda_n\,a_n.\cos(\lambda_n z).I_1(\lambda_n R)$$

$$a_n=\frac{\int_0^{L}\frac{\mu pR\omega}{k}\cos(\lambda_n z)dz}{\lambda_n I_1(\lambda_n R)\underbrace{\int_0^{l}\cos^2(\lambda_n z)dz}_{\frac{L}{2}}}=\frac{\mu pR\omega\lambda_n\sin(\lambda_n L)}{\lambda_n kI_1(\lambda_n R)\frac{L}{2}}$$

$$\Rightarrow a_n=\frac{2\mu pR\omega\sin(\lambda_n L)}{kLI_1(\lambda_n R)}$$

مساله ۲۹-۴)

برای دیسک اول:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)+\frac{d^2T}{dx^2}=0\,,\,BC\begin{cases}\dfrac{dT_1}{dr}(0,x)=0\\ -k\dfrac{dT_1}{dr}(R,x)=h_2T_1(R,x)\\ -k\dfrac{dT_1}{dx}(r,L)=h_1T_1(r,L)\end{cases}$$

شرایط مرزی در سطح مشترک
$$\begin{cases}T_1(0,r)=T_2(0,r)\\ -k\dfrac{dT_1}{dx}+\mu pr(\omega_1+\omega_2)=-k'\dfrac{dT_2}{dx}\end{cases}$$

$$T_1(r,x)=R_1(x).X_1(x)\Rightarrow\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda_n^2$$

$$\Rightarrow R_1(r)=A_{1n}J_0(r\lambda_n)+B_{2n}Y_0(\lambda_n r)$$

$$\frac{dR_1}{dr}(0)=0\Rightarrow B_n=0\Rightarrow -k\frac{dR_1}{dr}(R)=h_2R_1(R)$$

$\Rightarrow -k\dfrac{d}{dr}\left(J_0(R\lambda_n)\right)=h_2J_0(R\lambda_n)\Rightarrow$ λ_n حاصل خواهد شد

$$X''_1-\lambda_n^2X_1=0\Rightarrow X_1(x)=C_n\sinh(\lambda_n x)+D_n\cosh(\lambda_n x)$$

$$-k\frac{dX_1}{dx}(L)=h_1X_1(L)\Rightarrow -C_{1n}\lambda_n\cosh(\lambda_n L)-D_{1n}\lambda_n\sinh(\lambda_n L)=$$

$$\frac{h_1}{k}\left(C_{1n}\sinh(\lambda_n L)+D_{1n}\cosh(\lambda_n L)\right)$$

$$\Rightarrow D_{1n}=\underbrace{\frac{\lambda_n\cosh(\lambda_n L)+\frac{h_1}{k}\sinh(\lambda_n L)}{(-1)\left(\lambda_n\sinh\lambda_n L+\frac{h_1}{k}\cosh\lambda_n L\right)}}_{p_n}C_{1n}$$

$$T_1(r,x)=\sum a_n.J_0(r\lambda_n).\left[\sinh(\lambda_n x)+p_n\cosh(\lambda_n x)\right]$$

برای دیسک دوم:

$$T_2(r,x)=\sum b_n.J_0(r\mu_n)\left[\sinh(\mu_n x)+q_n\cosh(\mu_n x)\right]$$

$-\dfrac{d}{dr}\left(J_0(r\mu_n)\right)=\dfrac{h_2}{k}J_0(R\mu_n)\Rightarrow$ μ_n به دست خواهد آمد

$$q_n = \frac{\mu_n \cosh(\mu_n L') + \frac{h_1}{k'}\sinh(\mu_n L')}{(-1)\left(\mu_n \sinh(\mu_n L') + \frac{h_1}{k'}\cosh(\mu_n L')\right)}$$

$$T_1(r,0) = T_2(r,0) \Rightarrow \sum_{n=0}^{\infty} a_n . J_0(r\lambda_n).p_n = \sum_{n=0}^{\infty} b_n . J_0(r\mu_n).q_n$$

$$-k\frac{dT_1}{dx} + \mu r p(\omega_1+\omega_2) = -k'\frac{dT_2}{dx}, x = 0 \Rightarrow$$

$$+k\sum_{n=0}^{\infty} a_n . J_0(r\lambda_n).\lambda_n = +k'\sum_{n=0}^{\infty} b_n . J_0(r\mu_n).\mu_n - \mu r p(\omega_1+\omega_2)$$

با استفاده از این دو معادله a_n و b_n به دست خواهند آمد:

$$a_n = \frac{\int_0^R rJ_0(r\lambda_n)J_0(r\mu_n)\mu r p(\omega_1+\omega_2)dr}{\left(k'\int_0^r rJ_0^2(\mu_n r)\mu_n J_0(\lambda_n r)dr\right)\left(\frac{\int_0^R q_n r J_0^2(r\lambda_n)J_0(r\mu_n)dr}{\int_0^R p_n r J_0^2(r\mu_n)J_0(r\lambda_n)dr}\right) - \int_0^R rJ_0^2(r\lambda_n)J_0(r\mu_n)\lambda_n dr}$$

$$b_n = \frac{\int_0^R rJ_0(r\mu_n)\mu r p(\omega_1+\omega_2)dr}{\left(\int_0^R rJ_0^2(r\lambda_n)J_0(r\mu_n)\lambda_n dr\right) - \left(\frac{\int_0^R p_n r J_0(r\lambda_n)J_0^2(r\mu_n)dr}{\int_0^R q_n r J_0(r\mu_n)J_0^2(r\lambda_n)dr}\right)\left(\int_0^R rJ_0^2(r\lambda_n)J_0(r\mu_n)J_0(r\mu_n)\lambda_n dr\right)}$$

a) if $k = k'$, $L = L' \Rightarrow \mu_n = \lambda_n$, $q_n = p_n$, $a_n \neq b_n$

b) if $k = k'$, $L \neq L' \Rightarrow \mu_n = \lambda_n$, $q_n \neq p_n$, $a_n \neq b_n$

c) if $k \neq k'$, $L = L' \Rightarrow \mu_n \neq \lambda_n$, $q_n \neq p_n$, $a_n \neq b_n$

d) if $L' \ll L$, $k' \gg k \Rightarrow$

معادلات ساده‌تری خواهیم داشت زیرا دیسک دوم را متمرکز فرض می‌کنیم

e) if $h_1 = 0 \Rightarrow$ ساده‌تری خواهیم داشت p_n.

f) $\Rightarrow w_2 = 0$

مساله ۳۰-۴)

$$q_r.A|_r - q_r.A|_{r+dr} + q_z.S|_z - q_z.S|_{z+dz} = 0$$

$$-\frac{d}{dr}\left(-k2\pi r.dz + \frac{dT}{dr}\right)dr - \frac{d}{dr}(\rho c_p UT)dr$$

$$+\frac{d}{dz}\left(k2\pi r.dr\frac{dT}{dz}\right)dz = 0$$

$$\rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right) - \overbrace{\frac{\rho v c_p}{2\pi k}}^{\propto}\frac{dT}{dr} + r\frac{d^2T}{dz^2} = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) - \frac{\propto}{r}\frac{dT}{dr} + \frac{d^2T}{dz^2} = 0 , \theta = T - T_\infty$$

$$BC\begin{cases}\frac{dT}{dr}(0,z) = 0 \\ T(R,Z) = T_0 \\ -k\frac{dT}{dz}(r,0) = h_1(T(r,0) - T_\infty) \\ -k\frac{dT}{dz}(r,L) = h_2(T(r,L) - T_\infty)\end{cases} \rightarrow BC\begin{cases}\frac{d\theta}{dr}(0,z) = 0 \\ \theta(R,Z) = \theta_0 \\ -k\frac{d\theta}{dz}(r,0) = h\theta(r,0) \\ -k\frac{d\theta}{dz}(r,L) = h\theta(r,L)\end{cases}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - \frac{\propto}{r}\frac{d\theta}{dr} + \frac{d^2T}{dz^2} = 0$$

$$\theta(r,z) = \Psi(r,z) + \phi(r)$$

$$(I)\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) - \frac{\propto}{r}\frac{d\Psi}{dr} + \frac{d^2\Psi}{dz^2} = 0 , BC\begin{cases}\frac{d\Psi}{dr}(0,z) = 0 \\ \Psi(R,Z) = \theta_0 \\ -k\frac{d\Psi}{dr}(r,0) = h\Psi(r,0) \\ -k\frac{d\Psi}{dz}(r,z) = h\Psi(r,z)\end{cases}$$

$$(II)\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\propto}{r}\frac{d\phi}{dr} = 0 , BC\begin{cases}\frac{d\phi}{dr}(0) = 0 \\ \phi(R) = 0\end{cases}$$

برای معادله دوم:

$$r^2\frac{d^2\phi}{dr^2} + (1-\propto)r\frac{d\phi}{dr} = 0 , \frac{d\phi}{dr} = m$$

$$\rightarrow r^2m^2 + (1-\propto)rm = 0 \rightarrow \Delta = (1-\propto)m \Rightarrow \begin{cases}S_1 = \frac{(1-\propto)(m-1)}{2m^2} \\ S_2 = \frac{(1-\propto)(-m-1)}{2m^2}\end{cases}$$

$$\Rightarrow \phi(r) = C_n r^{\frac{(1-\propto)(m-1)}{2m^2}} + D_n r^{\frac{(1-\propto)(-m-1)}{2m^2}}$$

$$\frac{d\phi}{dr}=0\Rightarrow D_n=0$$

برای معادله اول:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right)-\frac{\propto}{r}\frac{d\Psi}{dr}+\frac{d^2\Psi}{dz^2}=0$$

$$\Psi(r,z)=R(r).Z(z)$$

$$\Rightarrow\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\frac{\propto}{rR}\frac{dR}{dr}=-\frac{d^2Z}{Zdz^2}=\lambda_n^2$$

$$\Rightarrow\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\frac{\propto}{r}\frac{dR}{dr}-\lambda_n^2R=0$$

$$\Rightarrow r^2\frac{d^2R}{dr^2}+(1-\alpha)r\frac{dR}{dr}-\lambda_n^2r^2R=0$$

$$v=\frac{1-(1-\alpha)}{2}=\frac{\alpha}{2}\Rightarrow R(r)=r^{\frac{\alpha}{2}}\left[A_nI_{\frac{\alpha}{2}}(\lambda r)+B_nk_{\frac{\alpha}{2}}(\lambda r)\right]$$

$$\frac{dR(0)}{dr}=0\Rightarrow B_n=0\Rightarrow R(r)=r^{\frac{\alpha}{2}}A_nI_{\frac{\alpha}{2}}(\lambda r)$$

$$\frac{d^2Z}{dz^2}+\lambda^2Z=0\Rightarrow Z(z)=E_n\sin(\lambda z)+F_n\cos(\lambda z)$$

$$-k\frac{dZ(0)}{dz}=h_1Z(0)\Rightarrow -k\lambda E_n=h_1F_n\Rightarrow E_n=\frac{h_1F_n}{-k\lambda}$$

$$\Rightarrow Z(z)=F_n\left[\frac{-h_1}{k\lambda}\sin(\lambda z)+\cos(\lambda z)\right]$$

$$\Rightarrow\Psi(r,z)=\sum_{n=0}^{\infty}a_nr^{\frac{\alpha}{2}}(\lambda r)I_{\frac{\alpha}{2}}(\lambda r)\left[-\frac{h_1}{k\lambda}\sin(\lambda z)+\cos(\lambda z)\right]$$

$$\theta_0=\sum_{n=0}^{\infty}a_nR^{\frac{\alpha}{2}}I_{\frac{\alpha}{2}}(\lambda r)\left[-\frac{h_1}{k\lambda}\sin(\lambda z)+\cos(\lambda z)\right]$$

$$\Rightarrow a_n=\frac{\theta_0\int_0^L\sin(\lambda z)\,dz}{R^{\frac{\alpha}{2}}I_{\frac{\alpha}{2}}(\lambda r)\int_0^L\left[-\frac{h_1}{k\lambda}\sin^2(\lambda z)+\sin(\lambda z)\cos(\lambda z)\right]dz}$$

مساله ۳۱-۴)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{d^2\theta}{dx^2}=0$$

$$BC\begin{cases}\left.\begin{matrix}\theta(R_i,x)=0\\ \theta(R_0,x)=0\end{matrix}\right\}\text{همگن}\\ \left.\begin{matrix}\theta(r,0)=\theta_0\\ \theta(r,\infty)=0\end{matrix}\right\}\text{ناهمگن}\end{cases}$$

$$\theta(r,x)=R(r).X(x)$$

$$\Rightarrow\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda_n^2$$

$$\Rightarrow\begin{cases}\frac{d^2X}{dx^2}-\lambda_n^2X=0\\ r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda_n^2r^2R=0\end{cases}$$

$$\Rightarrow\begin{cases}X(x)=C_ne^{-\lambda_nx}\Rightarrow X(x)=C_ne^{-\lambda_nx}\\ R(r)=A_nJ_0(\lambda_nr)+B_nY_0(\lambda_nr)\end{cases}$$

$$\begin{cases}R(R_i)=0\\ R(R_o)=0\end{cases}\Rightarrow$$

$$\begin{cases}A_nJ_0(\lambda_nR_i)+B_nY_0(\lambda_nR_i)=0\Rightarrow A_n=-B_n\frac{Y_0(\lambda_nR_i)}{J_0(\lambda_nR_i)}\\ A_nJ_0(\lambda_nR_0)+B_nY_0(\lambda_nR_i)=0\Rightarrow Y_0(\lambda_nR_i)J_0(\lambda_nR_0)=J_0(\lambda_nR_i)Y_0(\lambda_nR_o)\end{cases}$$

λ_n با استفاده از این معادله حاصل خواهد شد $\quad,\ n=0,1,2,\ldots$

$$\Rightarrow\theta(r,x)=\sum_{n=0}^{\infty}a_ne^{-\lambda_nx}\left[J_0(\lambda_nr)-\frac{J_0(\lambda_nR_i)}{Y_0(\lambda_nR_i)}Y_0(\lambda_nr)\right]$$

$$\Rightarrow\theta(r,0)=\theta_0=\sum_{n=0}^{\infty}a_n\left[J_0(\lambda_nr)-\frac{J_0(\lambda_nR_i)}{Y_0(\lambda_nR_i)}Y_0(\lambda_nr)\right]$$

$\Rightarrow a_n$ به دست خواهد آمد

مساله ۳۲-۴)

فرض می‌شود که سیب رسیده روی درخت یک کره با شعاع R است که حرارت را به صورت زیر از خورشید دریافت می‌کند:

$$q''(\phi) = \begin{cases} q''_0 \sin\phi \quad , & 0 < \phi < \pi \\ 0 \quad , & \pi < \phi < z\pi \end{cases}$$

فرمولاسیون کروی: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{1}{r^2 \sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dT}{d\phi}\right) = 0$

$$B.C: \begin{cases} (1)\ T(0,\phi) = finite \\ (2)\ \frac{dT}{dr}(R,\phi) = -\frac{h}{k}(T(R,\phi) - T_\infty) \quad \pi < \phi < 2\pi \\ (3)\ -k\frac{dT(R,\phi)}{dr} + q''_0 \sin\phi = h(T(R,\phi) - T_\infty) \quad \pi < \phi < \pi \\ \left.\begin{matrix} T(r,0) = finite \\ T(r,\pi) = finite \end{matrix}\right\} \text{همگن} \end{cases}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) + \frac{1}{r^2 \sin\phi}\frac{d}{d\phi}\left(\sin\phi\ \frac{d\theta}{d\phi}\right) = 0$$

$$\theta(r,\phi) = R(r).\phi(\phi) \Rightarrow \frac{1}{r^2}\frac{d}{dr}(r^2\phi R') + \frac{1}{r^2 \sin\phi}\frac{d}{d\phi}(R\phi'.\sin\phi) = 0$$

$$\frac{1}{R}\frac{d}{dr}(r^2 R') = -\frac{1}{\phi \sin\phi}\frac{d}{d\phi}(\phi'.\sin\phi) = +\lambda_n^2$$

$$\begin{cases} r^2 R'' + 2rR'' - \lambda_n^2 R = 0 \quad Euler \\ \phi'' + \frac{\cos\phi}{\sin\phi}\phi' + \lambda_n^2\phi = 0 \quad \text{logender} \Rightarrow \lambda_n^2 = n(n+1) \quad n = 0, \dots, \infty \end{cases}$$

$$x = \cos\phi \Rightarrow \phi(\phi) = A_n p_n(\cos\phi) + B_n q_n(\cos\phi)$$

$$T(r,0) = finite \Rightarrow \phi(0) = finite \Rightarrow \phi(\phi) = A_n p_n(\cos\phi)$$

$$R(r) = C_n r^n + D_n r^{-(n+1)}, n = -\frac{1}{2} + \left(\lambda_n + \frac{1}{4}\right)^{\frac{1}{2}}$$

$$(1) \Rightarrow D_n = 0 \Rightarrow R(r) = C_n r^n$$

$$\Rightarrow \theta(r,\phi) = \sum_{n=0}^{\infty} a_n r^n p_n(\cos\phi)$$

$$(3) \Rightarrow -k\sum_{n=0}^{\infty} a_n.R^{n-1}p_n(\cos\phi) + q''\sin\phi = h\sum_{n=0}^{\infty} a_n.R^n.p_n(\cos\phi)$$

$$\xRightarrow{\times p_n(\cos\phi)\sin\phi\, d\phi} -k\int_0^\pi a_n.nR^{n-1}p_n(\cos\phi)\sin\phi d\phi +$$

$$\int_0^\pi q''\sin^2\phi p_n(\cos\phi)d\phi = h\int_0^\pi a_n.R^n.p_n^2(\cos\phi).\sin\phi\, d\phi$$

$$\Rightarrow a_n = \frac{\int_0^\pi q''\sin^2\phi.p_n(\cos\phi)\, d\phi}{\int_0^\pi(-kR^{n-1} - hR^n)p_n^2(\cos\phi).\sin\phi\, d\phi}$$

مساله ۳۳-۴)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) + \frac{1}{r^2}\frac{d^2\theta}{d\varphi^2} = 0$$

$$BC \begin{cases} (I)\quad -k\frac{d\theta}{dr}(R_i,\theta) = f(\varphi) \\ (II)\quad -k\frac{d\theta}{dr}(R_0,\varphi) = h\theta(R_0,\varphi) \\ (III)\quad \theta(r,\varphi) = \theta(r,\varphi + 2\pi) \\ (IV)\quad \frac{d\theta}{r d\varphi}(r,\varphi) = \frac{d\theta}{r d\varphi}(r,\varphi + 2\pi) \end{cases}, f(\varphi) = \begin{cases} 0 & 0 < \varphi < \pi \\ q'' & \pi < \varphi < z\pi \end{cases}$$

$$\theta(r,\varphi) = R(r).T(\varphi) \Rightarrow \frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2 T}\frac{d^2T}{d\varphi^2} = 0$$

$$\frac{1}{T}\frac{d^2T}{d\varphi^2} = -\lambda_n^2 \Rightarrow T(\varphi) = C_{1n}\sin(\lambda_n\varphi) + C_{2n}\cos(\lambda_n\varphi)$$

$$(III) \Rightarrow C_{1n} = 0 \Rightarrow T(\varphi) = C_{2n}\cos(\lambda_n\varphi)$$

$$(IV) \Rightarrow \sin(\lambda_n\varphi) = 0 \Rightarrow \lambda_n = \frac{n\pi}{\varphi}, n = 1, 2, \dots$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda_n^2 = 0 \Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda_n^2 R = 0 \Rightarrow \begin{matrix} \alpha = 2 \\ \beta = 0 \end{matrix}$$

$$\Rightarrow \beta - \alpha + z = 0 \Rightarrow Caushy - Euler$$

$$\begin{cases} if \begin{array}{c}\lambda_n = 0 \\ (n=0)\end{array} \Rightarrow R_0(r) = -\frac{C_1}{r} + C_2 \overset{(II)}{\Rightarrow} R_0(r) = C_1\left[-\frac{1}{r} + \frac{C_2}{C_1}\right] \\ if \begin{array}{c}\lambda_n \neq 0 \\ n = 1,2,3,\ldots\end{array} \Rightarrow m^2 + m - \lambda_n^2 = 0 \Rightarrow m_{1,2} = \frac{-1 \pm \sqrt{1+4\lambda_n^2}}{2} \Rightarrow \end{cases}$$

$R(r) = A_n r^{m_1} + B_n r^{m_2}$

$(II) \Rightarrow -k\left[m_1 A_n R_0^{m_1-1} + m_2 B_n R_0^{m_2-1}\right] = h\left[A_n R_0^{m_1} + B_n R_0^{m_2}\right]$

$B_n = HA_n$

$\theta(r,\varphi) = a_0 R_0(r) + \sum_{n=1}^{\infty}[r^{m_1} + Hr^{m_2}][a_n \cos\lambda_n \varphi + b_n \sin\lambda_n \varphi]$,

$\lambda_n = n$

$(II) \Rightarrow -k\frac{d\theta}{dr}(R_i,\varphi) = \varphi(\phi)$

$\frac{d\theta}{dr}(r,\varphi) = \frac{a_0 C_1}{r^2} + \sum_{n=1}^{\infty}[m_1 r^{m_1-1} + m_2 H r^{m_2-1}][a_n \cos\lambda_n \varphi + b_n \sin\lambda_n \varphi]$

$f(\varphi) = \frac{ka.C_1}{R_i^2} + k\sum_{n=1}^{\infty}\left[m_1 R_i^{m_1-1} + m_2 H R_i^{m_2-1}\right]\left[a_n \cos\lambda_n \varphi + b_n \sin\lambda_n \varphi\right]$

$-\frac{ka.C_1}{R_i^2} = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)d\varphi = \frac{1}{2\pi}\int_0^{2\pi} q'' \, d\varphi = \frac{q''}{2} \Rightarrow a.C_1 = -\frac{q'' R_i^2}{2k}$

$-k\left[m_1 R_i^{m_1-1} + m_2 H R_i^{m_2-1}\right]a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\cos\varphi d\varphi = 0 \Rightarrow$

$a_n = 0 \Rightarrow k\left[m_1 R_i^{m_1-1} + m_2 H R_i^{m_2-1}\right]b_n =$

$\frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\cos\varphi \, d\varphi = \frac{q''}{2\pi}\int_0^{2\pi} \sin n\varphi d\varphi = \frac{q''(-1)^n}{2\pi n}$

$b_n = \frac{q''(-1)^n/2\pi n}{-k\left[m_1 R_i^{m_1-1} + m_2 H R_i^{m_2-1}\right]}$

$\theta(r,\varphi) = \frac{-q'' R_i^2}{2k}\left[-\frac{1}{r} + \frac{C_2}{C_1}\right] - \sum_{n=1}^{\infty}\frac{[r^{m_1} + Hr^{m_2}]q''(-1)^n/2\pi n}{-k\left[m_1 R_i^{m_1-1} + m_2 H R_i^{m_2-1}\right]}\sin n\varphi$

مساله ۳۴-۴)

$-\frac{d}{dx}(q_x dy.dz)dx - \frac{d}{dy}(q_y dx.dz)dy - \frac{d}{dz}(q_z dx.dy)dz = 0$

$\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} + \frac{d^2T}{dz^2} = 0 \, , \, \theta = T - T_\infty \Rightarrow \frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2} = 0$

$$BC\begin{cases} \theta(0,y,z) = \theta_0 & (1) \\ \theta(\infty,y,z) = 0 & (2) \\ \frac{d\theta}{dy}(x,0,z) = 0 & (3) \\ -k\frac{d\theta}{dy}(x,L/2,z) = h_3\theta(x,L/2,z) & (4) \\ -k\frac{d\theta}{dz}(x,y,0) = h_1\theta(x,y,0) & (5) \\ -k\frac{d\theta}{dz}(x,y,l) = h_2\theta(x,y,l) & (6) \end{cases}$$

$\theta(x,y,z) = X(x).Y(y).Z(z) \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

$\frac{Y''}{Y} = -\frac{Z''}{Z} - \frac{X''}{X} = -\lambda_n^2$

$Y'' + \lambda_n^2 Y = 0 \Rightarrow Y(y) = A_{1n}\sin(\lambda_n y) + B_{1n}\cos(\lambda_n y)$

$(3) \Rightarrow A_1 = 0 \Rightarrow Y(y) = B_1\cos(\lambda_n y)$

$(4) \Rightarrow k\lambda\sin(\lambda L/2) = h_3(\cos\lambda L/2) \Rightarrow \lambda_n = \frac{h_3}{k}\cot\frac{\lambda_n L}{2} \Rightarrow$

با حل این معادله λ_n به دست خواهد آمد

$\frac{Z''}{Z} = -\frac{X''}{X} - \lambda_n^2 = \mu_n^2 \Rightarrow Z'' + \mu_n^2 z = 0 \Rightarrow Z(z) = A_2\sin\mu_n z + B_2\cos\mu_n z$

$(5) \Rightarrow -k\frac{dz(0)}{dz} = h_1 z(0) \Rightarrow -kA_2\mu_n = h_1 B_2 \Rightarrow A_2 = -\frac{h_1 B_2}{\mu_n k}$

$(6) \Rightarrow -k\frac{dz(l)}{dz} = h_2 z(l) \Rightarrow -k[A_2\mu_n\cos\mu_n l - B_2\mu_n\sin\mu_n l] =$

$h_2[A_2\sin\mu_n l + B_2\cos\mu_n l]$

$-kB_2\left[\frac{-h_1}{\mu_n k}\cos\mu_n l - \mu_n\sin\mu_n l\right] = h_2 B_2\left[\frac{-h_1}{\mu_n k}\sin\mu_n l - \cos\mu_n l\right] \Rightarrow$

با حل این معادله μ_n به دست خواهد آمد

$\frac{X''}{X} = \lambda_n^2 + \mu_n^2 \Rightarrow X'' - (\lambda_n^2 + \mu_n^2)X = 0$

$$X(x) = C_n \exp\left(-\left(\lambda_n^2 + \mu_n^2\right)^{0.5} x\right)$$

$$\theta(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_n \cos(\lambda_n y)\left(\frac{-h_1}{\mu_m k}\sin\mu_n z + \cos\mu_n z\right) \times$$

$$\exp\left(-\left(\lambda_n^2 + \mu_n^2\right)^{0.5} x\right)$$

$(1) \Longrightarrow$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_n \cos(\lambda_n y)\left(\frac{-h_1}{\mu_n k}\sin\mu_n z + \cos\mu_n z\right) \xrightarrow{\times \cos\lambda_n y \sin\mu_n z \, dr\, dz}$$

$$A_n = \frac{\int_0^L\int_0^l \theta_0 \cos(\lambda_n y)\sin(\mu_n z)\, dy\, dz}{\int_0^L\int_0^l \cos^2(\lambda_n y)\left(\frac{-h_1}{\mu_n k}\sin^2(\mu_n z) + \cos(\mu_n z)\sin(\mu_n z)\right) dy\, dz}$$

مساله ۳۵-۴)

$$\frac{1}{r^2}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \frac{1}{r^2}\frac{d^2\theta}{d\theta^2} = 0$$

$$BC\begin{cases} \theta(0,z,\varphi) = finite \quad (1) \\ \frac{d\theta(R,z,\varphi)}{dr} = \frac{q''(\varphi)}{k} \begin{cases} q''(\varphi) = q'' & 0 < \varphi < \pi \\ q''(\varphi) = 0 & \pi < \varphi < z\pi \end{cases} \quad (2) \\ \theta(r,z,0) = \theta(r,z,2\pi) \quad (3) \\ \frac{d\theta(r,z,\varphi)}{d\varphi} = \frac{d\theta(r,z,2\pi)}{d\varphi} \quad (4) \\ \frac{d\theta(r,0,\varphi)}{dz} = \frac{h}{k}\theta(r,0,\varphi) \quad (5) \\ \frac{d\theta(r,L,\varphi)}{dz} = -\frac{h}{k}\theta(r,L,\varphi) \quad (6) \end{cases}$$

$$\theta = R(r).\phi(\varphi).Z(z) \Longrightarrow \frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\varphi r^2}\frac{d^2\phi}{d\varphi^2} = -\frac{1}{z}\frac{d^2 Z}{dz^2} = \lambda_n^2$$

$$\frac{d^2 Z}{dz^2} + \lambda_n^2 Z = 0 \Longrightarrow Z = A_n.\cos(\lambda_n z) + B_n.\sin(\lambda_n z)$$

$$(5) \Longrightarrow -\frac{h}{k\lambda_n}A_n + B_n = 0 \Longrightarrow B_n = \frac{h}{k\lambda_n}A_n$$

$(6) \Longrightarrow \tan(\lambda_n L) = \frac{2}{\frac{k\lambda_n}{h} + \frac{h}{k\lambda_n}}$ با حل این معادله حاصل خواهد شد λ_n

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda_n^2 r^2 = -\frac{1}{\varphi}\frac{d^2\phi}{d\varphi^2} = \gamma_m^2 \Longrightarrow \frac{d^2\phi}{d\varphi^2} + \gamma_m^2\varphi = 0$$

$$\Longrightarrow \phi(\varphi) = C_n\cos(\gamma_m\varphi) + D_n\sin(\gamma_m\varphi)$$

$$\begin{cases} \theta(r,z,0) = \theta(r,z,2\pi) \\ \frac{d\theta(r,z,0)}{d\varphi} = \frac{d\theta(r,z,2\pi)}{d\varphi} \end{cases} \Longrightarrow \gamma_m = n \text{ و } m = 1,2,3,\dots$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \left(\lambda_n^2 r^2 + \gamma_m^2\right)R = 0 \Longrightarrow R = E_m I_\gamma(\lambda_n r) + F_m K_\gamma(\lambda_n r)$$

$$\theta(0,z,\varphi) = finite \Longrightarrow F_m = 0 \Longrightarrow R = E_m I_\gamma(\lambda_n r)$$

$$\theta(r,z,\varphi) =$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_\gamma(\lambda_n r)\{A_n\cos(n\varphi) + B_n\sin(n\varphi)\}\{C_m\cos(\lambda_n z) +$$

$$D_n\sin(\lambda_n z)\}$$

$$=$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_n(\lambda_m r)\ \{A_{m,n}\cos(n\varphi)\cos(\lambda_m z) +$$

$$B_{m,n}\sin(n\varphi)\cos(\lambda_m z) + C_{m,n}\cos(n\varphi)\sin(\lambda_m z) +$$

$$D_{m,n}\sin(n\varphi)\sin(\lambda_m z)\}$$

$$\frac{d\theta(R,z,\varphi)}{dr} = \frac{q''(\varphi)}{k} \Longrightarrow$$

$$\frac{q''(\varphi)}{k} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{\lambda_m I_{n-1}(\lambda_m R)\}\ \{A_{m,n}\cos(n\varphi)\cos(\lambda_m z) +$$

$$B_{m,n}\sin(n\varphi)\cos(\lambda_m z) + C_{m,n}\cos(n\varphi)\sin(\lambda_m z) +$$

$$D_{m,n}\sin(n\varphi)\sin(\lambda_m z)\}$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} - m^2XY = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} - m^2 = 0$$

$$A_{m,n}=\frac{\int_0^L\int_0^{2\pi}\frac{q'''(\varphi)}{k}\cos(n\varphi)\cos(\lambda_m z)d\varphi dz}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}=$$

$$\frac{\int_0^L\int_0^{2\pi}\frac{q'''}{k}\cos(n\varphi)\cos(\lambda_m z)d\varphi dz}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}=0$$

$$B_{m,n}=\frac{\int_0^L\int_0^{2\pi}\frac{q'''(\varphi)}{k}\sin(n\varphi)\cos(\lambda_m z)d\varphi+\int_\pi^{2\pi}0.d\varphi\, dz}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}=$$

$$\frac{\frac{q'''}{kn\lambda_m}\sin(\lambda_m L)(1-(-1)^n)}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}$$

$$B_{m,n}=\frac{4q'''\sin(\lambda_m L)(1-(-1)^n)}{kn\pi^{\circ}\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\{\sin(\lambda_m L)\cos(\lambda_m L)+\lambda_m L\}}$$

$$C_{m,n}=\frac{\int_0^L\left(\int_0^{\pi}\frac{q'''(\varphi)}{k}\cos(n\varphi)\sin(\lambda_m z)d\varphi+\int_\pi^{2\pi}0.d\varphi\right)dz}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}=0$$

$$D_{m,n}=\frac{\int_0^L\left(\int_0^{\pi}\frac{q'''(\varphi)}{k}\sin(n\varphi)\sin(\lambda_m z)d\varphi+\int_\pi^{2\pi}0.d\varphi\right)dz}{\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\int_0^L\int_0^{2\pi}\cos(n\varphi)^2\cos(\lambda_m z)^2 d\varphi dz}=$$

$$\frac{\frac{q'''(\varphi)}{k}(1-\cos(\lambda_m L))\quad(1-(-1)^n)}{\frac{\pi}{4}\lambda_m\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\{\lambda_m L-\sin(\lambda_m L)\cos(\lambda_m L)\}}$$

$$D_{m,n}=\frac{4q'''(1-\cos(\lambda_m L))\quad(1-(-1)^n)}{kn\pi^{\circ}\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}\{\lambda_m L-\sin(\lambda_m L)\cos(\lambda_m L)\}}$$

$$\frac{\theta(r,z,\varphi)}{4q'''/k\pi}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-(-1)^n)\quad I_n(\lambda_m r)}{n.\left\{\lambda_m I_{n-1}(\lambda_m R)-\frac{n}{R}I_n(\lambda_m R)\right\}}.\frac{\sin(\lambda_m L).\sin(n\varphi)\cos(\lambda_m z)}{\sin(\lambda_m L)\cos(\lambda_m L)+\lambda_m L}-$$

$$\frac{(1-\cos(\lambda_m L)).\sin(n\varphi)\sin(\lambda_m z)}{\sin(\lambda_m L)\cos(\lambda_m L)-\lambda_m L}$$

ادامه حل به عهده خواننده گذاشته می‌شود.

فصل ۵

جداسازی متغیرها

مسائل ناپایا

مسائل ۱-۵ تا ۶-۵ در مورد سیستم‌های متمرکز است

مساله ۱-۵)

$$\dot{E}_{in}-\dot{E}_{out}=\frac{dE}{dt}\Longrightarrow q''A_1-hA_1(T-T_\infty)=\rho c\delta A_1\frac{dT}{dt}$$

$$T-T_\infty=\theta,\Longrightarrow q''-h\theta=\rho c\delta\frac{d\theta}{dt}\Longrightarrow\frac{d\theta}{dt}+\frac{h}{\rho c\delta}\theta=\frac{q''}{\rho c\delta},\quad\frac{h}{\rho c\delta}=m$$

$$\frac{d\theta}{dt}+m\theta=\frac{q''}{\rho c\delta}\Longrightarrow\ \theta=\frac{q''}{h}+Ce^{-mt}$$

$$t=0\Longrightarrow\theta=0\Longrightarrow\frac{q''}{h}+C=0\Longrightarrow C=-\frac{q''}{h}\Longrightarrow\theta=\frac{q''}{h}(1-e^{-mt})$$

مساله ۲-۵)

برای دیوار:

$$h_fA_1(T_f-T_w)-h_lA_2(T_w-T_\infty)-h_oA_3(T_w-T_\infty)=\rho_wc_wA_w2L\frac{dT_w}{dt}\ (I)$$

$$A_1=P.Vt,\ A_2=P(2L-Vt),\ A_3=P.2L$$

برای سیال:

$$-h_f P.dx(T_f - T_w) - \left(V\rho_f A_f c_f \frac{dT_f}{dx}\right)dx = \rho_f c_f A_f.dx.\frac{dT_f}{dt}$$

$$V = \frac{dx}{dt} \Rightarrow \frac{V}{dx} = \frac{1}{dt} \Rightarrow -h_f P.dx(T_f - T_w) = 2\rho_f c_f A_f.dx.\frac{dT_f}{dt} \quad (II)$$

با جایگذاری (II) درون (I) خواهیم داشت:

$$\frac{-2A_1}{\rho}\frac{dT_f}{dt} - \frac{h_i A_2}{(\rho cA)_f}(T_w - T_\infty) - \frac{h_o A_3}{(\rho cA)_f}(T_w - T_\infty) = \frac{(\rho cA)_w}{(\rho cA)_f}2L\frac{dT_w}{dt} \quad (III)$$

$$(\rho c_p A)_f \gg (\rho c_p A)_w \Rightarrow \frac{(\rho cA)_w}{(\rho cA)_f} \cong 0$$

از معادله (II) خواهیم داشت:

$$\frac{-2(\rho cA)_f}{h_f.\rho}\frac{dT_f}{dt} = \left((T_f - T_w) - (T_w - T_\infty)\right) \Rightarrow$$

$$\frac{2(\rho cA)_f}{h_f.\rho}\frac{dT_f}{dt} + (T_f - T_w) = (T_w - T_\infty) \quad (IV)$$

با جایگذاری (IV) درون (I) خواهیم داشت:

$$h_f A_1\left(\underbrace{(T_w - T_\infty) - \frac{2(\rho cA)_f}{h_f.\rho}\frac{dT_f}{dt}}_{(T_f - T_\infty)} - (T_w - T_\infty)\right) -$$

$$(h_i A_2 + h_o A_3)\underbrace{\left(\frac{2(\rho cA)_f}{h_f.\rho}\frac{dT_f}{dt} + (T_f - T_w)\right)}_{(T_w - T_\infty)} = \rho_w c_w A_w 2L\frac{dT_w}{dt}$$

$$\Rightarrow -h_f A_1\frac{2}{h_f.\rho}\cdot\frac{dT_f}{dt} - (h_i A_2 + h_o A_3)\left(\frac{2}{h_f.\rho}\frac{dT_f}{dt} + \frac{T_f - T_w}{(\rho cA)_f}\right) = \frac{2L(\rho cA)_w}{(\rho cA)_f}\cdot\frac{dT_w}{dt}$$

$$,\frac{(\rho cA)_w}{(\rho cA)_f} \cong 0$$

$$\Rightarrow -\frac{dT_f}{dt}\underbrace{\left(h_f A_1\frac{2}{h_f.\rho} + 2\left(\frac{h_i A_2 + h_o A_3}{h_f.\rho}\right)\right)}_{\alpha} - \underbrace{\left(\frac{h_i A_2 + h_o A_3}{(\rho cA)_f}\right)}_{\beta}T_f + \underbrace{\left(\frac{h_i A_2 + h_o A_3}{(\rho cA)_f}\right)}_{\beta}T_\infty =$$

$$0$$

معادله دیفرانسیل معمولی که حل آن بسیار ساده می‌باشد.

مساله ۳-۵)

برای دیوار:

$$-h_1(T_w - T_\infty) - h_2(T_w - T_f) = (\rho c\delta)_w\frac{dT_w}{dt}$$

$$-h_1(T_w - T_\infty) - h_2\left((T_w - T_\infty) - (T_f - T_\infty)\right) = (\rho c\delta)_w\frac{dT_w}{dt}$$

$$T_w - T_\infty = \theta_w, T_f - T_\infty = \theta_f$$

$$\Rightarrow -(h_1 + h_2)\theta_w - h_2\theta_f = (\rho c\delta)_w\frac{d\theta_w}{dt}, \theta_w(0) = \theta_0$$

برای سیال محدود:

$$-h_2(T_f - T_w) = (\rho c\delta)_f\frac{dT_f}{dt}$$

$$-h_2\left((T_f - T_\infty) - (T_w - T_\infty)\right) = (\rho c\delta)_f\frac{dT_f}{dt}$$

$$\Rightarrow h_2\theta_w - h_2\theta_f = (\rho c\delta)_f\frac{d\theta_f}{dt}, \theta_f(0) = \theta_0$$

$$\Rightarrow \begin{cases} \underbrace{\frac{-(h_1+h_2)}{(\rho c\delta)_w}}_{m_1}\theta_w + \underbrace{\frac{-h_2}{(\rho c\delta)_w}}_{m_2}\theta_f = \frac{d\theta_w}{dt} \\ \underbrace{\frac{h_2}{(\rho c\delta)_f}}_{m_3}\theta_w + \underbrace{\frac{-h_2}{(\rho c\delta)_f}}_{m_4}\theta_f = \frac{d\theta_f}{dt} \end{cases}$$

فرض: $\theta_w(t) = C_1 e^{\lambda t}$, $\theta_f(t) = C_2 e^{\lambda t}$

$$\Rightarrow \begin{cases} m_1 C_1 e^{\lambda t} + m_2 C_2 e^{\lambda t} = C_1\lambda e^{\lambda t} \\ m_3 C_1 e^{\lambda t} + m_4 C_2 e^{\lambda t} = C_2\lambda e^{\lambda t} \end{cases} \Rightarrow \begin{cases} m_1 C_1 + m_2 C_2 = C_1\lambda \\ m_3 C_1 + m_4 C_2 = C_2\lambda \end{cases}$$

$$\Rightarrow \begin{cases} (m_1 - \lambda)C_1 + m_2 C_2 = 0 \\ m_3 C_1 + (m_4 - \lambda)C_2 = 0 \end{cases} \Rightarrow \begin{vmatrix} (m_1 - \lambda) & m_2 \\ m_3 & (m_4 - \lambda) \end{vmatrix} = 0$$

$\Rightarrow (m_1 - \lambda)(m_4 - \lambda) - m_2 m_3 = 0 \Rightarrow$ بدست خواهند آمد λ_1, λ_2

$$\Rightarrow \begin{cases}\theta_w(t) = C_{11}e^{\lambda_1 t} + C_{12}e^{\lambda_2 t}\\ \theta_f(t) = C_{21}e^{\lambda_1 t} + C_{22}e^{\lambda_2 t}\end{cases} \Rightarrow \begin{cases}\theta_w(0) = \theta_0 = C_{11}e^{\lambda_1 0} + C_{12}e^{\lambda_2 0}\\ \theta_f(0) = \theta_0 = C_{21}e^{\lambda_1 0} + C_{22}e^{\lambda_2 0}\end{cases}$$

$$C_{11} = C_{21} = \frac{\theta_0+\theta_0}{2} = \theta_0, C_{12} = C_{22} = \frac{\theta_0-\theta_0}{2} = 0$$

$$\theta_w(t) = \theta_0 e^{\lambda_1 t}\,,\ \theta_f(t) = \theta_0 e^{\lambda_2 t}$$

مساله ۴-۵)

a)

$$\text{Shoe:} \underbrace{\frac{k_S}{k_S+k_D}\mu p \frac{D}{2}\omega(t)}_{\beta} - h_1 A \underbrace{(T_S - T_\infty)}_{\theta_S} = \underbrace{(\rho c)_S A\delta_1}_{\alpha}\frac{dT_S}{dt}$$

$$\text{Drum:} \underbrace{\frac{k_D}{k_S+k_D}\mu p \frac{D}{2}\omega(t)}_{\beta'} - h_2 A \underbrace{(T_D - T_\infty)}_{\theta_D} = \underbrace{(\rho c)_D A\delta_2}_{\alpha'}\frac{dT_D}{dt}$$

$$\Rightarrow \begin{cases}\beta - h_1 A\theta_S = \alpha\frac{d\theta_S}{dt}\\ \beta' - h_2 A\theta_D = \alpha'\frac{d\theta_D}{dt}\end{cases} \Rightarrow \begin{cases}\frac{\beta}{\alpha} - \frac{h_1 A}{\alpha}\theta_S = \frac{d\theta_S}{dt},\ \theta_S(0) = \theta_0\\ \frac{\beta'}{\alpha'} - \frac{h_2 A}{\alpha'}\theta_D = \frac{d\theta_D}{dt},\ \theta_D(0) = \theta_0\end{cases}$$

با این فرض که $\omega(t)$ یک مقدار مشخص و ثابت است معادله دیفرانسیل را حل می‌کنیم.

$$\theta_S = \left(\theta_0 + \frac{\beta}{h_1}\right)\exp\left(-\frac{h_1 t}{\alpha}\right) - \frac{\beta}{h_1}$$

$$\theta_D = \left(\theta_0 + \frac{\beta'}{h_2}\right)\exp\left(-\frac{h_2 t}{\alpha'}\right) - \frac{\beta'}{h_2}$$

مساله ۵-۵)

در هر دو سیال داخلی و خارجی از انتقال محوری صرفنظر می‌کنیم.

برای سیال خارجی:

$$-\rho_o V C_o \pi(R^2 - R_o^2)\frac{\partial\theta}{\partial x}dx + V(2\pi R_o dx)(\phi - \theta) = \rho_o C_o \pi(R^2 - R_o^2)dx\frac{\partial\theta}{\partial t}$$

$$\frac{1}{A_o u_o} = \frac{\ln\left(\frac{R_o}{R_i}\right)}{2k\pi dx} + \frac{1}{2\pi dx R_o h_o}\,,\ A_o = 2\pi R_o dx \Rightarrow \frac{1}{u_o} = \frac{R_o}{k}\ln\left(\frac{R_o}{R_i}\right) + \frac{1}{h_o}$$

$$\Rightarrow \frac{\partial\theta}{\partial x} - \frac{2R_o u_o}{\rho_o V C_o (R^2 - R_o^2)}(\phi - \theta) = -\frac{1}{V}\frac{\partial\theta}{\partial t}$$

فرض می‌کنیم دمای سیال خارجی به T_∞ برسند:

$$\theta(L,t) = T_\infty\,,\ \theta(x,0) = T_0$$

مدل برای دمای دیواره لوله:

$$k\pi(R^2 - R_o^2)\frac{\partial^2\phi}{\partial x^2}dx + U(2\pi R_i dx)(T - \phi) - \frac{2R_o V}{k(R^2 - R_o^2)}(\phi - \theta) = \frac{1}{\alpha}\frac{\partial\phi}{\partial t}$$

$$\phi(0,t) = T_0\,,\ \phi(L,t) = f(t)\,,\ \phi(x,0) = T_0$$

مسائل ۷-۵ تا ۳۲-۵ در مورد سیستم‌های توزیع‌شده می‌باشد.

مساله ۷-۵)

موازنه انرژی:

$$q_x + u_x = q_{x+dx} + u_{x+dx} + \frac{\partial}{\partial t}(q_x) + \frac{\partial}{\partial t}(u_x)$$

$$q_x + u_x = q_x + \frac{\partial}{\partial x}(q_x)dx + u_x + \frac{\partial}{\partial x}(u_x)dx + \frac{\partial}{\partial t}(q_x)dx + \frac{\partial}{\partial t}(u_x)dx$$

$\frac{\partial}{\partial x}(u_x) = 0$ صرفنظر از تغییر مکانی رسانایی

$\frac{\partial}{\partial t}(q_x) = 0$ صرفنظر از تغییر زمانی رسانایی

$$\Rightarrow \frac{\partial}{\partial x}(q_x)dx + \frac{\partial}{\partial t}(u_x)dx = 0\,,\ q_x = -k\frac{\partial T}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right)dx = -\frac{\partial}{\partial t}(\rho C_p T)dx\,,\ \alpha = \frac{k}{\rho C_p}$$

$$\Rightarrow \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

مساله ۸-۵)

a)

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{u'''}{\rho c}\,,\ \text{BC}\begin{cases}\frac{\partial T}{\partial x}(0,t) = 0\\ -k\frac{\partial T}{\partial x}(L,t) = h(T - T_\infty)\end{cases},\ \text{IC: } T(x,0) = T_\infty$$

$$\theta = T - T_\infty \Rightarrow \frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2} + \frac{u'''}{\rho c}\,,\ \text{BC}\begin{cases}\frac{\partial\theta}{\partial x}(0,t) = 0\\ -k\frac{\partial\theta}{\partial x}(L,t) = h\theta(L,t)\end{cases},$$

$$\text{IC: } \theta(x,0) = 0$$

$$\theta(x,y) = \psi(x,t) + \phi(x)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{u'''}{k}=0\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\phi}{\partial x}(0)=0\\ -k\frac{\partial\phi}{\partial x}(L)=h\phi(L)\end{cases}$$

$$\Rightarrow \phi(x)=\frac{u'''L^2}{2k}\left[\frac{2k}{hL}+1-\left(\frac{x}{L}\right)^2\right]=\frac{u'''L^2}{2k}\left[\frac{2}{Bi}+1-\left(\frac{x}{L}\right)^2\right]$$

$$\frac{\partial\psi}{\partial t}=\alpha\frac{\partial^2\psi}{\partial x^2}\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\psi}{\partial x}(0,t)=0\\ -k\frac{\partial\psi}{\partial x}(L,t)=h\psi(L,t)\end{cases}\ ,\ \mathrm{IC}:\psi(x,0)=-\phi(x)$$

$$\psi(x,t)=f(x).g(t)\Rightarrow f(x)=C_{1n}\sin(\lambda_n x)+C_{2n}\cos(\lambda_n x)$$

$$\frac{\partial f}{\partial x}(0)=0\Rightarrow C_{1n}=0\ ,\ kC_{2n}\,\lambda_n\sin(\lambda_n L)=h\,C_{2n}\cos(\lambda_n L)$$

λ_n به دست خواهد آمد

$$g(t)=C_{3n}.\exp(-\alpha\lambda_n^2 t)$$

$$\Rightarrow \psi(x,t)=\sum_{n=0}^{\infty}C_n.\cos(\lambda_n x).\exp(-\alpha\lambda_n^2 t)$$

$$\psi(x,0)=-\phi(x)=-\frac{u'''L^2}{2k}\left[\frac{2}{Bi}+1-\left(\frac{x}{L}\right)^2\right]=\sum_{n=0}^{\infty}C_n.\cos(\lambda_n x)$$

$$\Rightarrow C_n=\frac{\int_0^L\frac{-u'''L^2}{2k}\left[\frac{2}{Bi}+1-\left(\frac{x}{L}\right)^2\right]\cos(\lambda_n x).dx}{\int_0^L\cos^2(\lambda_n x).dx}=-\frac{2u'''L(-1)^n}{k\lambda_n}\left[\frac{1}{Bi}+\frac{1}{L^2\lambda_n^2}\right]$$

$$\Rightarrow \theta(x,y)=$$

$$\frac{u'''L^2}{2k}\left[\frac{2}{Bi}+1-\left(\frac{x}{L}\right)^2\right]-\sum_{n=0}^{\infty}\frac{2u'''L(-1)^n}{k\lambda_n}\left[\frac{1}{Bi}+\frac{1}{L^2\lambda_n^2}\right].\cos(\lambda_n x).\exp(-\alpha\lambda_n^2 t)$$

(b

فرض $\theta=a_2x^2+a_1x+a_0$

$$\frac{\partial\theta}{\partial x}=2a_2x+a_1\Rightarrow\frac{\partial^2\theta}{\partial x^2}=2a_2\ ,\ \frac{\partial\theta}{\partial x}(0,t)=0\Rightarrow a_1=0$$

$$-k\frac{\partial\theta}{\partial x}(L,t)=h\theta(L,t)\Rightarrow a_0=-a_2L^2\left(1+\frac{2}{Bi}\right)$$

$$\theta=a_2x^2-a_2L^2\left(1+\frac{2}{Bi}\right)\Rightarrow\theta=a_2L^2\left[\left(\frac{x}{L}\right)^2-\left(1+\frac{2}{Bi}\right)\right]$$

$$\Rightarrow\frac{\partial\theta}{\partial a_2}=L^2\left[\left(\frac{x}{L}\right)^2-\left(1+\frac{2}{Bi}\right)\right]\ ,\ \frac{1}{\alpha}\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}+\frac{u'''}{k}$$

$$\Rightarrow\int_0^L\frac{\partial^2\theta}{\partial x^2}dx+\int_0^L\frac{u'''}{k}dx=\frac{1}{\alpha}\int_0^L\frac{\partial\theta}{\partial t}dx\Rightarrow\left.\frac{\partial\theta}{\partial x}\right|_L-\left.\frac{\partial\theta}{\partial x}\right|_0+\frac{u'''L}{k}=$$

$$\frac{1}{\alpha}\frac{da_2}{dt}\int_0^L\frac{\partial\theta}{\partial a_2}dx\Rightarrow 2a_2L+\frac{u'''L}{k}=\frac{L^2}{\alpha}\frac{da_2}{dt}\int_0^L\left[\left(\frac{x}{L}\right)^2-\left(1+\frac{2}{Bi}\right)\right]dx=$$

$$\frac{L^2}{\alpha}\frac{da_2}{dt}\left[-2L\left(\frac{1}{Bi}+\frac{1}{3}\right)\right]\Rightarrow a_2=C\exp\left[\frac{-\alpha t}{L^2\left(\frac{1}{Bi}+\frac{1}{3}\right)}\right]-\frac{u'''}{2k}$$

$$\theta(x,0)=0\Rightarrow a_2=0\Rightarrow C=\frac{u'''}{2k}\Rightarrow a_2=\frac{u'''}{2k}\left(\exp\left[\frac{-\alpha t}{L^2\left(\frac{1}{Bi}+\frac{1}{3}\right)}\right]-1\right)$$

$$\theta(x,t)=\frac{u'''L^2}{2k}\left(\exp\left[\frac{-\alpha t}{L^2\left(\frac{1}{Bi}+\frac{1}{3}\right)}\right]-1\right)\left[\left(\frac{x}{L}\right)^2-\left(1+\frac{2}{Bi}\right)\right]$$

مساله ۹-۵)

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}\ ,\ \mathrm{BC}\begin{cases}-k\frac{\partial T}{\partial x}(L,t)+q''=h[T(L,t)-T_\infty]\\ +k\frac{\partial T}{\partial x}(0,t)=h[T(0,t)-T_\infty]\end{cases}\ ,\ \mathrm{IC}:T(x,t)=T_\infty$$

$$\theta=T-T_\infty\Rightarrow\frac{1}{\alpha}\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}\quad,\quad\mathrm{BC}\begin{cases}-k\frac{\partial\theta}{\partial x}(L,t)+q''=h\theta(L,t)\\ +k\frac{\partial\theta}{\partial x}(0,t)=h\theta(0,t)\end{cases}\ ,$$

$$\mathrm{IC}:\theta(x,t)=0$$

$$\theta(x,t)=\psi(x,t)+\phi(x)$$

$$\frac{\partial^2\phi}{\partial x^2}=0,\mathrm{BC}\begin{cases}-k\frac{\partial\phi}{\partial x}(L)+q''=h\phi(L)\\ +k\frac{\partial\phi}{\partial x}(0)=h\phi(0)\end{cases}\Rightarrow\ \phi(x)=\frac{q''}{2k+hL}x+\frac{kq''}{h(2k+hL)}$$

$$\frac{1}{\alpha}\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}\ ,\ \mathrm{BC}\begin{cases}-k\frac{\partial\psi}{\partial x}(L,t)=h\psi(L,t)\\ +k\frac{\partial\psi}{\partial x}(0,t)=h\psi(0,t)\end{cases}\ ,\ \mathrm{IC}:\psi(x,0)=-\phi(x)$$

$$\psi(x,t)=f(x).g(t)\Rightarrow f(x)=A_{1n}\sin(\lambda_n x)+A_{2n}\cos(\lambda_n x)$$

$$+k\frac{\partial f}{\partial x}(0)=hf(0)\Rightarrow A_{2n}=\frac{A_{1n}k\lambda_n}{h}$$

$-k\frac{\partial f}{\partial x}(L)=hf(L)\Rightarrow$ به دست خواهد آمد λ_n

$$f(x)=A_{1n}\left(\sin(\lambda_n x)+\frac{k\lambda_n}{h}\cos(\lambda_n x)\right)\ ,\ g(t)=A_{3n}.\exp(-\alpha\lambda_n^2 t)$$

$$\psi(x,t)=\sum_{n=0}^{\infty}A_n.\left(\sin(\lambda_n x)+\frac{k\lambda_n}{h}\cos(\lambda_n x)\right).\exp(-\alpha\lambda_n^2 t)$$

$$\psi(x,0)=-\left(\frac{q''}{2k+hL}x+\frac{kq''}{h(2k+hL)}\right)=\sum_{n=0}^{\infty}A_n.\left(\sin(\lambda_n x)+\frac{k\lambda_n}{h}\cos(\lambda_n x)\right)$$

مساله ۱۱-۵)

فرض می‌کنیم دمای اولیه برابر دمای محیط است و دمای پایه از T_∞ به T_0 تغییر می‌کند.

$$q_xA_x|_x-q_xA_x|_{x+dx}-A_sh(T-T_\infty)=\frac{\partial}{\partial t}(\rho A_x dx c_p T),h=\frac{h_1+h_2}{2}$$

$$A_x=\frac{bx}{L}l,A_s=Pdx,P=2l\ \rightarrow\ -\frac{\partial}{\partial x}(q_xA_x)-Ph(T-T_\infty)=$$

$$\rho c_p\frac{\partial}{\partial t}(A_xT)$$

$$q_x=-k\frac{\partial T}{\partial t}\rightarrow\ \frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right)-\frac{PLh}{lbk}(T-T_\infty)=\frac{\rho c_p}{k}\frac{\partial}{\partial t}(xT)$$

$$\rightarrow\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right)-m^2(T-T_\infty)=\frac{1}{\alpha}\frac{\partial}{\partial t}(xT)$$

$$\text{BC}\begin{cases}T(x,0)=T_\infty\ \rightarrow\theta(x,0)=0\\T(L,t)=T_0\ \rightarrow\theta(L,t)=\theta_0\\\frac{\partial T}{\partial x}(0,t)=0\rightarrow\frac{\partial\theta}{\partial x}(0,t)=0\end{cases}$$

$$\theta=T-T_\infty\ \rightarrow\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)-m^2\theta=\frac{1}{\alpha}x\frac{\partial\theta}{\partial t}\rightarrow\theta(x,t)=\psi(x,t)+\varphi(x)$$

$$\Rightarrow\frac{\partial}{\partial x}\left(x\frac{\partial\varphi}{\partial x}\right)-m^2\varphi=0\ ,\text{BC}\begin{cases}\varphi(L)=\theta_0\\\frac{\partial\varphi}{\partial x}(0)=0\end{cases}$$ از مساله ۱۱-۳:

$$\varphi(x)=\frac{\theta_0I_0(2mx^{0.5})}{I_0(2mL^{0.5})}$$

$$\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)-m^2\psi=\frac{1}{\alpha}x\frac{\partial\psi}{\partial t},\text{BC}\begin{cases}\psi(L,t)=0\\\frac{\partial\psi}{\partial x}(0,t)=0\end{cases}\quad\text{IC}:\psi(x,0)=-\varphi(x)$$

$$\Rightarrow\psi(x,t)=X(x).\tau(t)$$

$$\Rightarrow\ \tau(t)=\exp(-\lambda_n^2\alpha t)\ ,X(x)=C_1J_{2m}(2\lambda_nx^{0.5})+C_2\ J_{-2m}(2\lambda_nx^{0.5})$$

$$\frac{\partial X}{\partial x}(0)=0\rightarrow C_2=0\ ,X(L)=0\rightarrow C_1J_{2m}(2\lambda_nL^{0.5})=0$$

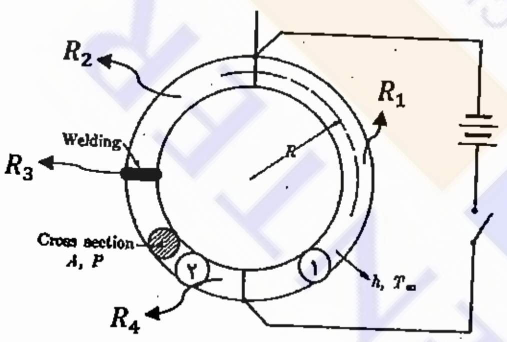
λ_n به دست خواهد آمد

$$\Rightarrow\psi(x,t)=X(x).\tau(t)=\sum_{n=0}^{\infty}A_nJ_{2m}\,(2\lambda_nx^{0.5})\exp(-\lambda_n^2\alpha t)$$

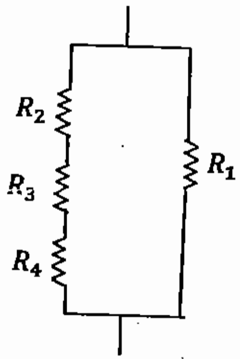
$$\text{From IC}: A_n=\frac{\int_0^L\left[-\frac{\theta_0I_0(2mx^{0.5})}{I_0(2mL^{0.5})}\right]x\,J_{2m}(2\lambda_nx^{0.5})dx}{\int_0^L x\,J_{2m}^2(2\lambda_nx^{0.5})dx}$$

$$\Rightarrow\theta(x,t)=\psi(x,t)+\varphi(x)$$

مساله ۱۲-۵)



یک مدار الکتریکی به صورت زیر را در نظر می‌گیریم



$$R_4+R_3+R_2>R_1$$
$$R_3\gg R_1$$

$$Z=R\theta,d\theta=-d\theta'$$

برای سیستم ۱:

$$-kA\frac{dT_1}{dz}\Big|_z-\left(-kA\frac{dT_1}{dz}\Big|_{z+dz}\right)-hPdz(T_1-T_\infty)+U_1'''Adz=\rho cAdz\frac{dT_1}{dt}$$

$$\Longrightarrow\frac{d^2T_1}{dz^2}-\frac{hP}{kA}(T_1-T_\infty)+\frac{U_1'''}{k}=\frac{1}{\alpha}\frac{dT_1}{dt}\ ,Z=R\theta$$

$$\Longrightarrow\frac{d^2T_1}{d\theta^2}-\frac{hPR^2}{kA}(T_1-T_\infty)+\frac{U_1'''R^2}{k}=\frac{R^2}{\alpha}\frac{dT_1}{dt}\ ,m^2=\frac{hPR^2}{kA}$$

$$T_1-T_\infty=U_1\Longrightarrow\frac{d^2U_1}{d\theta^2}-m^2U_1+\frac{U_1'''R^2}{k}=\frac{R^2}{\alpha}\frac{dU_1}{dt}$$

$$\text{BC}\begin{cases}U_1(0,t)=U_2(0,t)\\U_1(\pi,t)=U_2(\pi,t)\end{cases},\text{IC}:U_1(\theta,0)=0$$

$$U_1(\theta,t)=\psi_1(\theta,t)+\tau_1(\theta)$$

$$\frac{d^2\psi_1}{d\theta^2} - m^2\psi_1 = \frac{R^2}{\alpha}\frac{d\psi_1}{dt} \;,\; \text{BC}\begin{cases}\psi_1(0,t) = \psi_2(0,t) = 0\\ \psi_1(\pi,t) = \psi_2(\pi,t) = 0\end{cases} ,\; \text{IC:}\psi_1(\theta,0) = -\tau_1(\theta)$$

و

$$\frac{d^2\tau_1}{d\theta^2} - m^2\tau_1 + \frac{U_1'''R^2}{k} = 0 \,,\, \text{BC}\begin{cases}\tau_1(0) = \tau_2(0)\\ \tau_1(\pi) = \tau_2(\pi)\end{cases}$$

$$\Rightarrow \tau_1(\theta) = C_{1n}\sinh(m\theta) + C_{2n}\cosh(m\theta) + \frac{U_1'''A}{hP}$$

برای سیستم ۲:

$$T_2 - T_\infty = U_2 \Rightarrow \frac{d^2U_2}{d\theta^2} - m^2U_2 + \frac{U_2'''R^2}{k} = \frac{R^2}{\alpha}\frac{dU_2}{dt}$$

$$\text{BC}\begin{cases}U_1'(0,t) = -U_2'(0,t)\\ U_1'(\pi,t) = -U_2'(\pi,t)\end{cases} ,\text{IC: } U_2(\theta',0) = 0$$

$$U_2(\theta',t) = \psi_2(\theta',t) + \tau_2(\theta')$$

$$\frac{d^2\psi_2}{d\theta'^2} - m^2\psi_2 = \frac{R^2}{\alpha}\frac{d\psi_2}{dt} \;,\; \text{BC}\begin{cases}\psi_1'(0,t) = -\psi_2'(0,t) = 0\\ \psi_1'(\pi,t) = -\psi_2'(\pi,t) = 0\end{cases} ,\; \text{IC:}\psi_2(\theta',0) = -\tau_2(\theta')$$

و

$$\frac{d^2\tau_2}{d\theta'^2} - m^2\tau_2 + \frac{U_2'''R^2}{k} = 0 \,,\, \text{BC}\begin{cases}\tau_1'(0) = -\tau_2'(0) \Rightarrow C_{1n} = -C_{3n}\\ \tau_1'(\pi) = -\tau_2'(\pi) \Rightarrow C_{2n} = -C_{4n}\end{cases}$$

$$\Rightarrow \tau_2(\theta) = C_{3n}\sinh(m\theta) + C_{4n}\cosh(m\theta) + \frac{U_2'''A}{hP}$$

$$\tau_1(0) = \tau_2(0) \Rightarrow C_{2n} = \frac{(U_2''' - U_1''')A}{2hP}$$

$$\tau_1(\pi) = \tau_2(\pi) \Rightarrow C_{1n} = -\frac{1}{2}\frac{\frac{(U_2'''-U_1''')A}{2hP}[\cosh(m\pi)-1]}{\sinh(m\pi)}$$

$$\psi_1(\theta,t) = F_1(\theta).G_1(t) \Rightarrow F_1(\theta) = A_{1n}\sin(\beta_n\theta) + A_{2n}\cos(\beta_n\theta)$$

$$F_1(0) = 0 \Rightarrow A_{2n} = 0 \,,\, F_1(\pi) = 0 \Rightarrow \beta_n\pi = n\pi \Rightarrow \beta_n = n$$

$$G_1(t) = A_{3n}\exp\left(-\frac{\alpha\lambda_n^2}{R^2}t\right)$$

$$\Rightarrow \psi_1(\theta,t) = \sum_{n=1}^{\infty} A_n.\sin(n\theta).\exp\left(-\frac{\alpha\lambda_n^2}{R^2}t\right)$$

$$\psi_1(\theta,0) = -\tau_1(\theta) \Rightarrow A_n = \frac{-\int_0^\pi \tau_1(\theta).\sin(n\theta).d\theta}{\int_0^\pi \sin^2(n\theta).d\theta} = -\frac{2}{\pi}\int_0^\pi \tau_1(\theta).\sin(n\theta).d\theta$$

$$\psi_2(\theta',t) = F_2(\theta').G_2(t) \Rightarrow F_2(\theta') = B_{1n}\sin(\beta_n\theta') + B_{2n}\cos(\beta_n\theta')$$

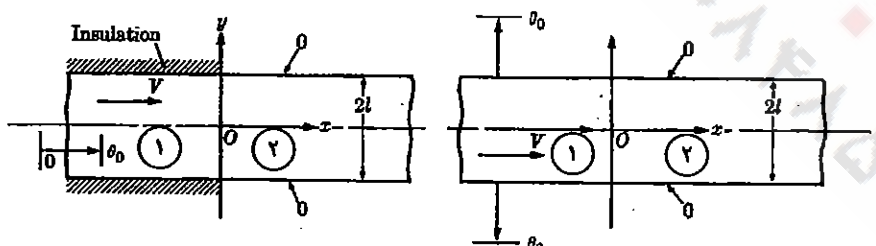
$$F_2'(0) = 0 \Rightarrow B_{1n} = 0 \,,\, F_2'(\pi) = 0 \Rightarrow \beta_n\pi = n\pi \Rightarrow \beta_n = n$$

$$G_2(t) = B_{3n}\exp\left(-\frac{\alpha\lambda_n^2}{R^2}t\right)$$

$$\Rightarrow \psi_2(\theta',t) = \sum_{n=1}^{\infty} B_n.\cos(n\theta').\exp\left(-\frac{\alpha\lambda_n^2}{R^2}t\right)$$

$$\psi_2(\theta',0) = -\tau_2(\theta') \Rightarrow B_n = \frac{-\int_0^\pi \tau_2(\theta').\cos(n\theta').d\theta'}{\int_0^\pi \cos^2(n\theta').d\theta'} = -\frac{2}{\pi}\int_0^\pi \tau_2(\theta').\cos(n\theta).d\theta$$

مساله ۱۳-۵



(a) (b)

(a

$$\frac{\partial^2\theta_1}{\partial y^2} - V\frac{\partial\theta_1}{\partial x} = \frac{1}{\alpha}\frac{\partial\theta_1}{\partial t} \,,\, \text{BC}\begin{cases}\theta_1(-\infty,y,t) = \theta_0\\ \frac{\partial\theta_1}{\partial y}(x,l,t) = 0\\ \frac{\partial\theta_1}{\partial y}(x,0,t) = 0\end{cases} ,\; \text{IC: } \theta_1(x,y,0) = 0$$

از آنجایی که عایق وجود دارد: $\frac{\partial^2\theta_1}{\partial y^2} = 0$

$$\frac{\partial^2\theta_2}{\partial y^2}-V\frac{\partial\theta_2}{\partial x}=\frac{1}{\alpha}\frac{\partial\theta_2}{\partial t}\quad,\quad \text{BC}\begin{cases}\theta_2(x,l,t)=0\\ \theta_2(0,y,t)=\theta_1(0,y,t)=\theta_0\\ \frac{\partial\theta_2}{\partial y}(x,0,t)=0\end{cases},$$

$$\text{IC: }\theta_2(x,y,0)=0$$

(b

$$\frac{\partial^2\theta_1}{\partial y^2}-V\frac{\partial\theta_1}{\partial x}=\frac{1}{\alpha}\frac{\partial\theta_1}{\partial t},\ \text{BC}\begin{cases}\theta_1(x,l,t)=\theta_0\\ \theta_1(0,y,t)=\theta_2(0,y,t)\\ \frac{\partial\theta_1}{\partial y}(x,0,t)=0\end{cases},\ \text{IC: }\theta_1(x,y,0)=0$$

$$\frac{\partial^2\theta_2}{\partial y^2}-V\frac{\partial\theta_2}{\partial x}=\frac{1}{\alpha}\frac{\partial\theta_2}{\partial t}\quad,\quad \text{BC}\begin{cases}\theta_2(x,l,t)=0\\ \theta_2(0,y,t)=\theta_1(0,y,t)\\ \frac{\partial\theta_2}{\partial y}(x,0,t)=0\end{cases},$$

$$\text{IC: }\theta_2(x,y,0)=0$$

حل معادلات به دانشجویان واگذار می‌شود.

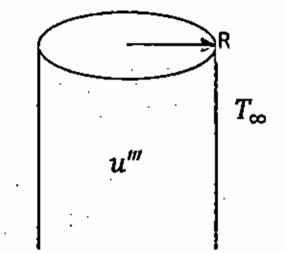
مساله ۱۵-۵)

حل مجدد مساله ۴-۵ (با ضریب انتقال حرارت بزرگ) برای یک استوانه جامد طولانی با شعاع R:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{u'''}{k}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t},\ \theta=T-T_\infty$$

$$\text{BC}\begin{cases}\theta(R,t)=0\\ \frac{\partial\theta}{\partial r}(0,t)=0\end{cases},\ \text{IC: }\theta(r,0)=0$$

$h\to\infty$



$$\theta(r,t)=\psi(r,t)+\phi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{u'''}{k}=0,\ \text{BC}\begin{cases}\phi(R)=0\\ \frac{\partial\phi}{\partial r}(0)=0\end{cases}\Rightarrow\phi(r)=\frac{-u'''r^2}{4k}+C_1\ln r+C_2$$

$$\Rightarrow\phi(r)=\frac{-u'''}{4k}(r^2-R^2)$$

و

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial\psi}{\partial t},\ \text{BC}\begin{cases}\psi(R,t)=0\\ \frac{\partial\psi}{\partial r}(0,t)=0\end{cases},\ \text{IC: }\psi(r,0)=-\phi(r)$$

$$\psi(r,t)=\omega(r).\tau(t)\Rightarrow\omega(r)=C_{1n}J_0(\lambda_n r)+C_{2n}Y_0(\lambda_n r)$$

$\omega(R)=0\Rightarrow C_{1n}J_0(\lambda_n R)=0\Rightarrow$ λ_n به دست خواهد آمد

$$\frac{\partial\omega}{\partial r}(0)=0\Rightarrow C_{2n}=0$$

$$\tau(t)=C_{3n}.\exp(-\lambda_n^2\alpha t)$$

$$\Rightarrow\psi(r,t)=\sum_{n=0}^{\infty}A_n.J_0(\lambda_n r).\exp(-\lambda_n^2\alpha t)$$

$$\psi(r,0)=-\phi(r)=\frac{u'''}{4k}(r^2-R^2)=\sum_{n=0}^{\infty}A_n.J_0(\lambda_n r)$$

$$\Rightarrow A_n=\frac{\int_0^R\frac{u'''}{4k}(r^2-R^2)rJ_0(\lambda_n r)dr}{\int_0^R rJ_0^2(\lambda_n r)dr=\frac{R^2}{2}J_1^2(\lambda_n R)}$$

$$\Rightarrow\theta(r,t)=T(r,t)-T_\infty=\sum_{n=0}^{\infty}A_n.J_0(\lambda_n r).\exp(-\lambda_n^2\alpha t)-\frac{u'''}{4k}(r^2-R^2)$$

حل مجدد مساله (a) ۸-۵ (با ضریب انتقال حرارت متوسط) برای یک استوانه جامد طولانی با شعاع R:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{u'''}{k}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t},\ \theta=T-T_\infty$$

$$\text{BC}\begin{cases}\frac{\partial\theta}{\partial r}(R,t)=\frac{h\theta}{k}\\ \frac{\partial\theta}{\partial r}(0,t)=0\end{cases},\ \text{IC: }\theta(r,0)=0$$

$$\theta(r,t)=\psi(r,t)+\phi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{u'''}{k}=0,\ \text{BC}\begin{cases}\frac{\partial\phi}{\partial r}(R)=\frac{h\phi}{k}\\ \frac{\partial\phi}{\partial r}(0)=0\end{cases}\Rightarrow\phi(r)=\frac{-u'''r^2}{4k}+C_1\ln r+C_2$$

$$\Rightarrow\phi(r)=\frac{-u'''}{4k}r^2+\frac{hu'''R^2-2u'''Rk}{4kh}$$

و

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial\psi}{\partial t},\ \text{BC}\begin{cases}\frac{\partial\psi}{\partial r}(R,t)=\frac{h\psi}{k}\\ \frac{\partial\psi}{\partial r}(0,t)=0\end{cases},\ \text{IC: }\psi(r,0)=-\phi(r)$$

$$\psi(r,t)=\omega(r).\tau(t)\Rightarrow\omega(r)=C_{1n}J_0(\lambda_n r)+C_{2n}Y_0(\lambda_n r)$$

$$\frac{\partial\omega}{\partial r}(R)=\frac{h\omega(R)}{k}\Rightarrow C_{1n}\lambda_n J_1(\lambda_n R)=\frac{h}{k}C_{1n}J_0(\lambda_n R)$$

$\Rightarrow$ λ_n به دست خواهد آمد

$$\frac{\partial\omega}{\partial r}(0)=0\Rightarrow C_{2n}=0$$

$$\tau(t) = C_{3n}.\exp(-\lambda_n^2 \alpha t)$$

$$\Rightarrow \psi(r,t) = \sum_{n=0}^{\infty} A_n.J_0(\lambda_n r).\exp(-\lambda_n^2 \alpha t)$$

$$\psi(r,0) = -\phi(r) = \frac{-u'''}{4k} r^2 + \frac{hu''' R^2 - 2u''' Rk}{4kh} = \sum_{n=0}^{\infty} A_n.J_0(\lambda_n r)$$

$$\Rightarrow A_n = \frac{\int_0^R \left[\frac{-u'''}{4k} r^2 + \frac{hu''' R^2 - 2u''' Rk}{4kh}\right] r J_0(\lambda_n r) dr}{\int_0^R r J_0^2(\lambda_n r) dr = \frac{R^2}{2} J_1^2(\lambda_n R)}$$

$$\Rightarrow \theta(r,t) = T(r,t) - T_\infty = \sum_{n=0}^{\infty} A_n.J_0(\lambda_n r).\exp(-\lambda_n^2 \alpha t) - \frac{u'''}{4k} r^2 + \frac{hu''' R^2 - 2u''' Rk}{4kh}$$

مساله ۱۶-۵)

$$q_r.A|_{r+dr} - q_r.A|_r = \rho c V \frac{\partial T}{\partial t}, A_r = 2\pi r l$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \text{BC}\begin{cases}\frac{\partial T}{\partial r}(0,t) = 0\\ k\frac{\partial T}{\partial r}(R,t) = q''\end{cases}, \text{IC: } T(r,0) = T_\infty$$

$$\theta = T - T_\infty \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \text{BC}\begin{cases}\frac{\partial \theta}{\partial r}(0,t) = 0\\ k\frac{\partial \theta}{\partial r}(R,t) = q''\end{cases},$$

IC: $\theta(r,0) = T_\infty$

$\theta(r,t) = \psi(r,t) + \phi(r) + \varphi(t)$

وقتی‌که $t \to \infty$ دما به صورت بینهایت زیاد می‌شود. برای نشان دادن این اثر از عبارت $\varphi(t)$ استفاده نمودیم.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t} = cte \ , \text{BC}\begin{cases}\frac{\partial \phi}{\partial r}(0) = 0\\ k\frac{\partial \phi}{\partial r}(R) = q''\end{cases}$$

$$\frac{1}{\alpha}\frac{\partial \varphi}{\partial t} = c \Rightarrow \varphi(t) = c\alpha t + c_1$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = c \Rightarrow \phi(r) = \frac{cr^2}{4} + c_2 \ln r + c_3$$

$$\text{At} = 0 \to c_2 = 0 , k\frac{\partial \phi}{\partial r}(R) = q'' \to c = \frac{2q''}{kR}$$

به طور اختیاری c_1 و c_3 را صفر فرض می‌کنیم

$$\Rightarrow \phi(r) = \frac{q'' r^2}{2kR}, \varphi(t) = \frac{2q'' t}{\rho c R}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \psi}{\partial t}, \text{BC}\begin{cases}\frac{\partial \psi}{\partial r}(0,t) = 0\\ \frac{\partial \psi}{\partial r}(R,t) = 0\end{cases}, \text{IC: } \psi(r,0) = -\phi(r) - \varphi(0)$$

$$\psi(r,t) = R(r).\tau(t) \Rightarrow R(r) = A_n J_0(\lambda_n r) + B_n Y_0(\lambda_n r)$$

$$\frac{\partial \psi}{\partial r}(0,t) = 0 \Rightarrow B_n = 0 \Rightarrow R(r) = A_n J_0(\lambda_n r) , \tau(t) = C_n \exp(-\alpha \lambda_n^2 t)$$

$$\psi(r,t) = \sum_{n=0}^{\infty} D_n.J_0(\lambda_n r).\exp(-\alpha \lambda_n^2 t)$$

$$\psi(r,0) = -\phi(r) - \varphi(0) == -\phi(r) \Rightarrow D_n = \frac{\int_0^R -\phi(r).r J_0(\lambda_n r) dr}{\int_0^R r J_0^2(\lambda_n r) dr = \frac{R^2}{2} J_1^2(\lambda_n R)}$$

$$\Rightarrow \theta(r,t) = T(r,t) - T_\infty = \sum_{n=0}^{\infty} D_n.J_0(\lambda_n r).\exp(-\lambda_n^2 \alpha t) + \frac{q'' r^2}{2kR} + \frac{2q'' t}{\rho c R}$$

مساله ۱۷-۵)

با توجه به مساله ۳۰-۳ کل حرارت تولیدی= $Pr\omega\mu = q_1 + q_2$

$$q_1 = Pr\omega\mu \frac{k_1/\delta_1}{k_1/\delta_1 + k_2/\delta_2}, q_2 = Pr\omega\mu \frac{k_2/\delta_2}{k_1/\delta_1 + k_2/\delta_2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) - \frac{h_1}{k\delta_1}\theta + \frac{P\omega\mu B}{\delta_1} r = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \frac{k_1/\delta_1}{k_1/\delta_1 + k_2/\delta_2} = B$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) - m^2\theta + nr = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \text{BC}\begin{cases}\frac{\partial \theta(0,t)}{\partial r} = 0\\ \frac{\partial \theta(R,t)}{\partial r} = \frac{h_3 \theta}{k}\end{cases}, \text{IC: } \theta(r,0) = 0$$

$$\theta(r,t) = \psi(r,t) + \phi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - m^2\psi = \frac{1}{\alpha}\frac{\partial \psi}{\partial t}, \text{BC}\begin{cases}\frac{\partial \psi(0,t)}{\partial r} = 0\\ \frac{\partial \psi(R,t)}{\partial r} = \frac{h_3 \psi}{k}\end{cases}, \text{IC: } \psi(r,0) = -\phi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) - m^2\phi + nr = 0 , \text{BC}\begin{cases}\frac{\partial \phi(0)}{\partial r} = 0\\ \frac{\partial \phi(R)}{\partial r} = \frac{h_3 \phi}{k}\end{cases}$$

حل معادلات به دانشجویان واگذار می‌شود.

مساله ۱۸-۵)

$$qA|_r - qA|_{r+dr} - hA'(T-T_\infty) = \rho c V \frac{\partial T}{\partial t},$$

$$A_r = 2\pi r\delta \,, A' = 2\pi r dr \,, V = 2\pi r\delta dr$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - m^2(T-T_\infty) = \frac{\rho c}{k}\frac{\partial T}{\partial t},$$

$$\text{BC}\begin{cases} T(0,t) = finite \\ T(R_i,t) = T_0 \end{cases}, \text{IC:}T(r,0) = T_\infty\,,$$

$$\theta = T - T_\infty \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) - m^2\theta = \frac{\rho c}{k}\frac{\partial \theta}{\partial t}$$

$$\text{BC}\begin{cases} \theta(0,t) = finite \\ \theta(R_i,t) = \theta_0 \end{cases}, \text{IC:}\theta(r,0) = 0$$

$$\theta(r,t) = \psi(r,t) + \phi(r):$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) - m^2\phi = 0\,, \text{BC}\begin{cases} \frac{\partial \phi}{\partial r}(0) = 0 \\ \phi(R_i) = \theta_0 \end{cases}$$

$$\phi(r) = A_n I_0(mr) + B_n K_0(mr)$$

$$\frac{\partial \phi}{\partial r}(0) = 0 \Rightarrow B_n = 0 \quad , \quad \phi(R_i) = \theta_0 \Rightarrow A_n = \frac{\theta_0}{I_0(mR_i)} \Rightarrow \phi(r) = \theta_0\frac{I_0(mr)}{I_0(mR_i)}$$

و

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - m^2\psi = \frac{1}{\alpha}\frac{\partial \psi}{\partial t} \quad , \quad \text{BC}\begin{cases} \psi(0,t) = finite \text{ or } \frac{\partial \psi}{\partial r}(0,t) = 0 \\ \psi(R_i,t) = 0 \end{cases},$$

$$\text{IC:}\psi(r,0) = -\phi(r)$$

$$\psi(r,t) = R(r).\tau(t) \Rightarrow R(r) = C_{1n}J_0(\lambda_n r) + C_{2n}Y_0(\lambda_n r)$$

$$R(0) = finite \Rightarrow C_{2n} = 0 \quad , \quad R(R_i) = 0 \Rightarrow C_{1n}J_0(\lambda_n R_i) = 0 \Rightarrow \lambda_n$$

$$\tau(t) = C_{3n}\exp(-\alpha(\lambda_n^2 + m^2)t)$$

$$\psi(r,t) = \sum_{n=0}^{\infty} B_n . J_0(\lambda_n r).\exp(-\alpha(\lambda_n^2 + m^2)t)$$

$$\psi(r,0) = -\phi(r) \Rightarrow B_n = \frac{-\theta_0}{I_0(mR_i)}\frac{\int_0^{R_i} r I_0(mr).J_0(\lambda_n r)dr}{\int_0^{R_i} r.J_0^2(\lambda_n r)dr}$$

$$\theta(r,t) = \sum_{n=0}^{\infty} B_n . J_0(\lambda_n r).\exp(-\alpha(\lambda_n^2 + m^2)t) + \theta_0\frac{I_0(mr)}{I_0(mR_i)}$$

مساله ۱۹-۵)

از توزیع دما در جهت r صرفنظر می‌کنیم.

$$\frac{\partial^2\theta}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \text{BC}\begin{cases} \frac{\partial \theta}{\partial z}(0,t) = -\frac{q_2''}{k} \\ \frac{\partial \theta}{\partial z}(H,t) = \frac{q_1''}{k} \end{cases}, \text{IC: }(z,0) = 0\,, \theta = \theta_1 + \theta_2$$

$$\frac{\partial^2\theta_1}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta_1}{\partial t}, \text{BC}\begin{cases} \frac{\partial \theta_1}{\partial z}(0,t) = 0 \\ \frac{\partial \theta_1}{\partial z}(H,t) = \frac{q_1''}{k} \end{cases}, \text{IC: }\theta_1(z,0) = 0$$

$$\frac{\partial^2\theta_2}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta_2}{\partial t}, \text{BC}\begin{cases} \frac{\partial \theta_2}{\partial z}(0,t) = -\frac{q_2''}{k} \\ \frac{\partial \theta_2}{\partial z}(H,t) = 0 \end{cases}, \text{IC: }\theta_2(z,0) = 0$$

$$\theta_1(z,t) = \psi_1(z,t) + \phi_1(z) + \varphi_1(t)$$

$$\frac{\partial^2\phi_1}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \varphi_1}{\partial t} = C \Rightarrow \varphi_1(t) = C\alpha t + C_1 \text{ و } \phi_1(z) = \frac{C}{2}z^2 + C_2 z + C_3$$

$$\text{BC}\begin{cases} \frac{\partial \phi_1}{\partial z}(0,t) = 0 \Rightarrow C_2 = 0 \\ \frac{\partial \phi_1}{\partial z}(H,t) = \frac{q_1''}{k} \Rightarrow C = \frac{q_1''}{Hk} \end{cases} \text{ فرض } C_3 = C_1 = 0$$

$$\Rightarrow \varphi_1(t) = \frac{q_1''}{Hk}\alpha t\,, \phi_1(z) = \frac{q_1''}{2Hk}z^2$$

$$\frac{\partial^2\psi_1}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \psi_1}{\partial t} \quad , \quad \text{BC}\begin{cases} \frac{\partial \psi_1}{\partial z}(0,t) = 0 \\ \frac{\partial \psi_1}{\partial z}(H,t) = 0 \end{cases} \quad , \quad \psi_1(z,0) = -\phi_1(z) - \varphi_1(0) = -\frac{q_1''}{2Hk}z^2$$

$$\psi_1(z,t) = Z_1(z).\tau_1(t) \Rightarrow Z_1(z) = A_{1n}\sin(\lambda_n z) + A_{2n}\cos(\lambda_n z)$$

$$\tau_1(t) = A_{3n}\exp(-\alpha\lambda_n^2 t)$$

$$\frac{\partial Z_1}{\partial z}(0) = 0 \Rightarrow A_{1n} = 0\,, \frac{\partial Z_1}{\partial z}(H) = 0 \Rightarrow -A_{2n}\lambda_n\sin(\lambda_n H) = 0$$

$$\Rightarrow \lambda_n = \frac{n\pi}{H}$$

$$\psi_1(z,t) = \sum_{n=0}^{\infty} A_n\cos(\lambda_n z).\exp(-\alpha\lambda_n^2 t)$$

$$\psi_1(z,0) = -\frac{q_1''}{2Hk}z^2 = \sum_{n=0}^{\infty} A_n \cos(\lambda_n z) \Longrightarrow A_n = \frac{\int_0^H -\frac{q_1''}{2Hk}z^2 .\cos(\lambda_n z)dz}{\int_0^H \cos^2(\lambda_n z)dz}$$

$$\theta_1(z,t) = \sum_{n=0}^{\infty} A_n \cos(\lambda_n z) . \exp(-\alpha\lambda_n^2 t) + \frac{q_1''}{2Hk}z^2 + \frac{q_1''}{Hk}\alpha t$$

و با روندی مشابه:

$$\theta_2(z,t) = \sum_{n=0}^{\infty} B_n \cos(\lambda_n z) . \exp(-\alpha\lambda_n^2 t) + \frac{q_2''}{2Hk}z^2 + \frac{q_2''}{Hk}\alpha t$$

$$\Longrightarrow \theta = \theta_1 + \theta_2$$

مساله ۲۰-۵)

از آنجایی‌که طول لوله زیاد است، انتقال حرارت را فقط در جهت r در نظر می‌گیریم.

$$qA|_r - qA|_{r+dr} + u'''(2\pi r dr L) = \frac{\partial}{\partial t}(mcT_1)$$

$$\Longrightarrow -\frac{\partial}{\partial r}(q_r . A_r)dr + u'''(2\pi r dr L) = \rho . 2\pi r L c . dr . \frac{\partial T}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{u'''}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

برای داخل راکتور (Rod) :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) + \frac{u'''}{k_1} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t}, \mathrm{BC}\begin{cases}\frac{\partial T_1}{\partial r}(0,t) = 0\\ k_1\frac{\partial T_1(R,t)}{\partial r} = k_2\frac{\partial T_2(R,t)}{\partial r}\end{cases}, \mathrm{IC}{:}T_1(r,0) = T_\infty$$

$$\theta_1 = T_1 - T_\infty \Longrightarrow$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_1}{\partial r}\right) + \frac{u'''}{k_1} = \frac{1}{\alpha_1}\frac{\partial \theta_1}{\partial t}, \mathrm{BC}\begin{cases}\frac{\partial \theta_1}{\partial r}(0,t) = 0\\ k_1\frac{\partial \theta_1(R,t)}{\partial r} = k_2\frac{\partial \theta_2(R,t)}{\partial r}\end{cases}, \mathrm{IC}{:}\theta_1(r,0) = 0$$

فرض: $\theta_1(r,t) = C(A_0 + A_1 r + r^2)\ ,\ C \neq 0$

$$\frac{\partial \theta_1}{\partial r}(0,t) = 0 \Longrightarrow A_1 = 0 \Longrightarrow \theta_1(r,t) = C(A_0 + r^2)$$

برای پوشش:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial \theta_2}{\partial t}, \mathrm{BC}\begin{cases}\theta_1(R,t) = \theta_2(R,t)\\ -k_2\frac{\partial \theta_2(R_0,t)}{\partial r} = h\theta_2(R_0,t)\end{cases}, \mathrm{IC}{:}\theta_2(r,0) = 0$$

فرض: $\theta_2(r,t) = C'(B_0 + B_1 r + r^2)\ ,\ C' \neq 0$

$$\theta_1(R,t) = \theta_2(R,t) \Longrightarrow B_1 = 0$$

$$-k_2\frac{\partial \theta_2(R_0,t)}{\partial r} = h\theta_2(R_0,t) \Longrightarrow B_0 = \frac{-k_2(2R_0)}{h} - R_0^2$$

$$k_1\frac{\partial \theta_1(R,t)}{\partial r} = k_2\frac{\partial \theta_2(R,t)}{\partial r} \Longrightarrow C' = \frac{k_1}{k_2}C$$

$$\theta_1(R,t) = \theta_2(R,t) \Longrightarrow A_0 = \frac{k_1}{k_2}(B_0 + R^2) - R^2 = \frac{k_1}{k_2}\left(\frac{-k_2(2R_0)}{h} - R_0^2 + R^2\right) - R^2$$

بنابراین:

$$\int_0^R \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_1}{\partial r}\right) + \frac{u'''}{k_1}\right]dr = \int_0^R \frac{1}{\alpha_1}\frac{\partial \theta_1}{\partial t}dr = \frac{1}{\alpha_1}\int_0^R \frac{\partial \theta_1}{\partial c}.\frac{dc}{dt}dr$$

$$\int_0^R \left[2C + \frac{1}{r}2Cr + \frac{u'''}{k_1}\right]dr = \frac{1}{\alpha_1}\frac{dc}{dt}\int_0^R (A_0 + r^2)dr$$

$$4CR + \frac{u'''}{k_1}R = \frac{1}{\alpha_1}\frac{dc}{dt}.\left(A_0 R + \frac{R^3}{3}\right) \Longrightarrow \left(\frac{12\alpha_1}{3A_0 + R^2}\right)C + \frac{3\alpha_1 u'''}{k_1(3A_0 + R^2)} = \frac{dc}{dt}$$

$$\Longrightarrow \frac{dc}{dt} - PC = q \Longrightarrow C(t) = \frac{q}{P}(1 - e^{-Pt}) = \frac{u'''}{4k_1}(e^{-Pt} - 1)\ ,\ C'(t) = \frac{u'''}{4k_2}(e^{-Pt} - 1)$$

برای میله: $\theta_1(r,t) = \frac{u'''}{4k_1}(e^{-Pt} - 1)\left(\frac{k_1}{k_2}\left(\frac{-k_2(2R_0)}{h} - R_0^2 + R^2\right) - R^2 + r^2\right)$

برای پوشش: $\theta_2(r,t) = \frac{u'''}{4k_2}(e^{-Pt} - 1)\left(\frac{-k_2(2R_0)}{h} - R_0^2 + r^2\right)$

مساله ۲۱-۵)

حل مجدد مساله ۴-۵ (با ضریب انتقال حرارت بزرگ) برای یک کره جامد با شعاع R:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) + \frac{u'''}{k} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \theta = T - T_\infty$$

$$\mathrm{BC}\begin{cases}\theta(R,t) = 0\\ \frac{\partial \theta}{\partial r}(0,t) = 0 \ or\ \theta(0,t) = finite\end{cases}, \mathrm{IC}{:}\ \theta(r,0) = 0$$

$$\theta(r,t) = \psi(r,t) + \phi(r)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{u'''}{k} = 0, \mathrm{BC}\begin{cases}\phi(R) = 0\\ \frac{\partial \phi}{\partial r}(0) = 0\end{cases} \Longrightarrow \phi(r) = \frac{-u'''r^2}{6k} - \frac{C_1}{r} + C_2$$

$$\Longrightarrow \phi(r) = \frac{-u'''}{6k}(r^2 - R^2)$$

و

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial\psi}{\partial t}\,,\ \mathrm{BC}\begin{cases}\psi(R,t)=0\\ \frac{\partial\psi}{\partial r}(0,t)=0\end{cases},\ \mathrm{IC}{:}\ \psi(r,0)=-\phi(r)$$

$$\psi(r,t)=\omega(r).\tau(t)\Rightarrow\omega(r)=C_{1n}J_{\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}+C_{2n}J_{-\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}$$

یا

$$\omega(r)=C'_{1n}\frac{\sin(\lambda_n r)}{r}+C'_{2n}\frac{\cos(\lambda_n r)}{r}\,,\ \frac{\partial\omega}{\partial r}(0)=0\Rightarrow C'_{2n}=0\,,$$

$$\omega(R)=0\Rightarrow C'_{1n}\frac{\sin(\lambda_n R)}{R}=0\Rightarrow\lambda_n=\frac{n\pi}{R}$$

$$\tau(t)=C_{3n}.\exp(-\lambda_n^2\alpha t)\Rightarrow\psi(r,t)=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\lambda_n^2\alpha t)$$

$$\psi(r,0)=-\phi(r)=\frac{u'''}{6k}(r^2-R^2)=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}$$

$$\Rightarrow A_n=\frac{\int_0^R\frac{u'''}{6k}(r^2-R^2)\sin(\lambda_n r)dr}{\int_0^R\frac{\sin^2(\lambda_n r)}{r}dr}$$

$$\Rightarrow\theta(r,t)=T(r,t)-T_\infty=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\lambda_n^2\alpha t)-\frac{u'''}{6k}(r^2-R^2)$$

حل مجدد مساله (a) ۸-۵ (با ضریب انتقال حرارت متوسط) برای یک کره جامد با شعاع R:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)+\frac{u'''}{k}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}\,,\ \theta=T-T_\infty$$

$$\mathrm{BC}\begin{cases}\frac{\partial\theta}{\partial r}(R,t)=\frac{h\theta}{k}\\ \frac{\partial\theta}{\partial r}(0,t)=0\end{cases},\ \mathrm{IC}{:}\ \theta(r,0)=0$$

$$\theta(r,t)=\psi(r,t)+\phi(r)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{u'''}{k}=0\,,\ \mathrm{BC}\begin{cases}\frac{\partial\phi}{\partial r}(R)=\frac{h\phi}{k}\\ \frac{\partial\phi}{\partial r}(0)=0\end{cases}\Rightarrow\phi(r)=\frac{-u'''r^2}{6k}-\frac{C_1}{r}+C_2$$

$$\Rightarrow\phi(r)=\frac{-u'''}{6k}r^2+\frac{hu'''R^2-2u'''Rk}{6kh}$$

و

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial\psi}{\partial t}\,,\ \mathrm{BC}\begin{cases}\frac{\partial\psi}{\partial r}(R,t)=\frac{h\psi}{k}\\ \frac{\partial\psi}{\partial r}(0,t)=0\end{cases},\ \mathrm{IC}{:}\ \psi(r,0)=-\phi(r)$$

$$\psi(r,t)=\omega(r).\tau(t)\Rightarrow\omega(r)=C_{1n}J_{\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}+C_{2n}J_{-\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}$$

یا

$$\omega(r)=C'_{1n}\frac{\sin(\lambda_n r)}{r}+C'_{2n}\frac{\cos(\lambda_n r)}{r}\,,\ \frac{\partial\omega}{\partial r}(0)=0\Rightarrow C'_{2n}=0$$

$\frac{\partial\omega}{\partial r}(R)=\frac{h\omega(R)}{k}\Rightarrow$. λ_n به دست خواهد آمد

$$\tau(t)=C_{3n}.\exp(-\lambda_n^2\alpha t)$$

$$\Rightarrow\psi(r,t)=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\lambda_n^2\alpha t)$$

$$\psi(r,0)=-\phi(r)=\frac{u'''}{6k}r^2-\frac{hu'''R^2-2u'''Rk}{6kh}=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}$$

$$\Rightarrow A_n=\frac{\int_0^R\left[\frac{u'''}{6k}r^2-\frac{hu'''R^2-2u'''Rk}{6kh}\right]\sin(\lambda_n r)dr}{\int_0^R\frac{\sin^2(\lambda_n r)}{r}dr}$$

$$\Rightarrow\theta(r,t)=T(r,t)-T_\infty=\textstyle\sum_{n=0}^{\infty}A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\lambda_n^2\alpha t)-\frac{u'''}{6k}r^2+\frac{hu'''R^2-2u'''Rk}{6kh}$$

مساله ۲۲-۵)

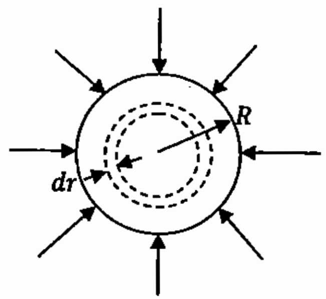
$$q.A|_{r+dr}-q.A|_r=\rho Vc\frac{dT}{dt}\,,\ A_r=4\pi r^2$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}\,,\ \mathrm{BC}\begin{cases}\frac{\partial T}{\partial r}(0,t)=0\\ \frac{\partial T}{\partial r}(R,t)=\frac{q''}{k}\end{cases}$$

IC: $(r,0)=T_\infty\,,\ \theta=T-T_\infty$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}\,,\ \mathrm{BC}\begin{cases}\frac{\partial\theta}{\partial r}(0,t)=0\\ \frac{\partial\theta}{\partial r}(R,t)=\frac{q''}{k}\end{cases},$$

IC: $\theta(r,0)=0$



$$\theta(r,t)=\psi(r,t)+\phi(r)+\varphi(t)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=C\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\phi}{\partial r}(0)=0\\ \frac{\partial\phi}{\partial r}(R)=\frac{q''}{k}\end{cases}$$

$$\phi(r)=\frac{Cr^2}{6}-\frac{C_1}{r}+C_2\ ,\ \frac{\partial\phi}{\partial r}(0)=0\Longrightarrow C_1=0$$

فرض می‌کنیم $C_2=0$

$$\frac{\partial\phi}{\partial r}(R)=\frac{q''}{k}\Longrightarrow\phi(r)=\frac{q''r^2}{2kR}$$

$$\frac{1}{\alpha}\frac{d\varphi}{dt}=C\ \Longrightarrow\varphi(t)=C\alpha t+C_3\ ,\ \varphi(0)=0\Rightarrow C_3=0\Rightarrow\varphi(t)=\frac{3q''\alpha}{kR}t$$

$$\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial r^2}=\frac{1}{\alpha}\frac{\partial\psi}{\partial t}\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\psi}{\partial r}(0,t)=0\\ \frac{\partial\psi}{\partial r}(R,t)=0\end{cases},\ \mathrm{IC}\!:\psi(r,0)=-\phi(r)-\varphi(0)$$

$$\psi(r,t)=f(r).g(t)\Longrightarrow f(r)=C_{1n}J_{\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}+C_{2n}J_{-\frac{1}{2}}(\lambda_n r)r^{-\frac{1}{2}}$$

$$\text{Or } f(r)=C'_{1n}\frac{\sin(\lambda_n r)}{r}+C'_{2n}\frac{\cos(\lambda_n r)}{r}\ \text{and}\ g(t)=C_{3n}\exp(-\lambda_n^2\alpha t)$$

$$f(0)=finite\Longrightarrow C'_{2n}=0\Longrightarrow f(r)=C'_{1n}\frac{\sin(\lambda_n r)}{r}$$

$$f'(R)=0\Longrightarrow\sin(\lambda_n R)=0\Rightarrow\lambda_n=\frac{(2n+1)\pi}{2R}$$

$$\psi(r,t)=\sum_{n=0}^{\infty}A_n\sin\left(\frac{(2n+1)\pi}{2R}r\right)\exp(-\lambda_n^2\alpha t)$$

$$\psi(r,0)=-\frac{q''r^2}{2kR}\Rightarrow A_n=\frac{\int_0^R\frac{q''r^2}{2kR}\sin(\lambda_n r)dr}{\int_0^R\sin^2(\lambda_n r)dr}$$

$$\theta(r,t)=\sum_{n=0}^{\infty}A_n\sin\left(\frac{(2n+1)\pi}{2R}r\right)\exp(-\lambda_n^2\alpha t)+\frac{q''r^2}{2kR}+\frac{3q''\alpha}{kR}t$$

مساله ۲۳-۵)

$$\theta'=T-T_\infty,\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta'}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\theta'}{\partial\theta}\right)=\frac{1}{\alpha}\frac{\partial\theta'}{\partial t}$$

$$\mathrm{BC}\begin{cases}\theta'(0,\theta,t)=finite\\ q''-h\theta'=k\frac{\partial\theta'}{\partial r}(R,\theta,t)\\ \frac{\partial\theta'}{\partial\theta}(r,0,t)=0\\ \frac{\partial\theta'}{\partial\theta}(r,\pi,t)=0\end{cases},\ \mathrm{IC}\!:\theta'(r,\theta,0)=\theta_0$$

$$\theta'(r,\theta,t)=\psi(r,\theta,t)+\phi(r,\theta)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0\ \mathrm{BC}\begin{cases}\phi(0,\theta)=finite\\ q''-h\phi=k\frac{\partial\phi}{\partial r}(R,\theta)\\ \frac{\partial\phi}{\partial\theta}(r,0)=0\\ \frac{\partial\phi}{\partial\theta}(r,\pi)=0\end{cases}$$

$$\phi(r,\theta)=f(r).g(\theta)\Rightarrow\frac{r^2f''+2rf'}{f}=-\frac{g''}{g}-\frac{\cos\theta}{\sin\theta}\frac{g'}{g}=\beta_n$$

$$\Rightarrow f(r)=C_{1n}r^{m_1}+C_{2n}r^{m_2}\ ,\ m_{1,2}=\frac{-1\pm\sqrt{1+4\beta_n}}{2}$$

$$g(\theta)=C_{3n}P_n(\cos\theta)+C_{4n}Q_n(\cos\theta)$$

$$\frac{\partial g}{\partial\theta}(0)=0,\frac{\partial g}{\partial\theta}(\pi)=0\Rightarrow C_{4n}=0\ ,\ \beta_n=n(n+1)\Rightarrow$$

$$m_1=n,m_2=-n\Rightarrow f(r)=C_{1n}r^n+C_{2n}r^{-n}$$

$$f(0)=finite\Rightarrow C_{2n}=0\Rightarrow\phi(r,\theta)=\sum_{n=0}^{\infty}C_nr^nP_n(\cos\theta)$$

$$\Rightarrow q''-h[C_nR^nP_n(\cos\theta)]=k[nC_nR^{n-1}P_n(\cos\theta)]$$

$$\Rightarrow q''=C_nR^n\left(h+\frac{kh}{R}\right)P_n(\cos\theta)\xrightarrow{\times P_n(\cos\theta)}C_n=\frac{\int_0^{\pi}q''P_n(\cos\theta)d\theta}{\int_0^{\pi}R^n\left(h+\frac{kh}{R}\right)P_n^2(\cos\theta)d\theta}\ ,$$

و

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)=\frac{1}{\alpha}\frac{\partial\psi}{\partial t}$$

$$\mathrm{BC}\begin{cases}\psi(0,\theta,t)=finite\\ -h\psi(R,\theta,t)=k\frac{\partial\psi}{\partial r}(R,\theta,t)\\ \frac{\partial\psi}{\partial\theta}(r,0,t)=0\\ \frac{\partial\psi}{\partial\theta}(r,\pi,t)=0\end{cases},\ \mathrm{IC}\!:\psi(r,\theta,0)=\theta_0-\phi(r,\theta)$$

$$\psi(r,\theta,t) = \omega(r).\varphi(\theta).\tau(t)$$

$$\frac{r^2\omega'' + 2r\omega'}{\omega} = -\frac{\varphi''}{\varphi} - \frac{\cos\theta}{\sin\theta}\frac{\varphi'}{\varphi} + \frac{1}{\alpha}\frac{\tau'}{\tau} = \lambda_n$$

$$\omega(r) = A_{1n}r^{s_1} + A_{2n}r^{s_2}\ ,\ s_{1,2} = \frac{-1 \pm \sqrt{1+4\lambda_n}}{2}$$

$$-\frac{\varphi''}{\varphi} - \frac{\cos\theta}{\sin\theta}\frac{\varphi'}{\varphi} = \lambda_n - \frac{1}{\alpha}\frac{\tau'}{\tau} = \mu_n \Rightarrow \varphi(\theta) = A_{3n}P_n(\cos\theta) + A_{4n}Q_n(\cos\theta)$$

$$\tau(t) = A_{5n}\exp\big((\lambda_n - \mu_n)\alpha t\big)\ ,\ \mu_n = n(n+1)\ n = 1, \dots, \infty$$

$$\omega(0) = finite \Rightarrow A_{2n} = 0$$

$$-h\omega(R) = k\frac{\partial\omega}{\partial r}(R) \Rightarrow -hR = ks_1 = k\frac{-1+\sqrt{1+4\lambda_n}}{2}$$

$$\Rightarrow \lambda_n = \left(\frac{1}{4} - \frac{2hR}{4k}\right)^2 - \frac{1}{4}, \frac{\partial\varphi}{\partial\theta}(0) = 0, \frac{\partial\varphi}{\partial\theta}(\pi) = 0 \Rightarrow A_{4n} = 0$$

$$\psi(r,\theta,t) = \sum_{n=0}^{\infty} A_n.P_n(\cos\theta).r^{s_1}.\exp\big((\lambda_n - \mu_n)\alpha t\big)$$

$$\psi(r,\theta,0) = \theta_0 - \phi(r,\theta) = \sum_{n=0}^{\infty} A_n.P_n(\cos\theta).r^{s_1} =$$

$$\theta_0 - \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta)$$

ادامه حل و به دست آوردن ثوابت به خواننده واگذار می‌شود.

مسئله ۲۴-۵)

For LOX: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t}$, BC$\begin{cases}\frac{\partial T_1}{\partial r}(0,t) = 0\\ T_1(R^*,t) = T_2(R^*,t)\end{cases}$,

IC: $T_1(r,0) = T_g$, $R^* =$ LOX شعاع کره.

For GOX: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t}$, BC$\begin{cases}\frac{\partial T_2}{\partial r}(R,t) = \frac{q''}{k}\\ T_1(R^*,t) = T_2(R^*,t)\\ \frac{\partial T_1}{\partial r}(R^*,t) = \frac{\partial T_2}{\partial r}(R^*,t)\end{cases}$,

IC: $T_2(r,0) = T_g$

با فرض اینکه از ضخامت لایه اکسیژن گازی صرفنظر کنیم:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \text{BC}\begin{cases}\frac{\partial T}{\partial r}(R,t) = \frac{q''}{k}\\ \frac{\partial T}{\partial r}(0,t) = 0\end{cases}, \text{IC: } T(r,0) = T_g$$

$$\theta = T - T_g \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}, \text{BC}\begin{cases}\frac{\partial\theta}{\partial r}(R,t) = \frac{q''}{k}\\ \frac{\partial\theta}{\partial r}(0,t) = 0\end{cases}, \theta(r,0) = 0$$

$$\theta(r,t) = \psi(r,t) + \phi(r) + \varphi(t)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\varphi}{\partial t} = C, \text{BC}\begin{cases}\frac{\partial\phi}{\partial r}(R) = \frac{q''}{k}\\ \frac{\partial\phi}{\partial r}(0) = 0\end{cases}, \varphi(0) = 0$$

$$\varphi(t) = C\alpha t + C_1 \Rightarrow \varphi(0) = 0 \Rightarrow C_1 = 0 \Rightarrow \varphi(t) = C\alpha t$$

$$\phi(r) = \frac{Cr^2}{6} - \frac{C_2}{r} + C_3, \frac{\partial\phi}{\partial r}(0) = 0 \Rightarrow C_2 = 0$$

فرض به صورت اختیاری $C_3 = 0$ ، $\frac{\partial\phi}{\partial r}(R) = \frac{q''}{k} \Rightarrow C = \frac{3q''}{kR} \Rightarrow \varphi(t) = \frac{3q''t}{\rho c_p R}$ ،

$$\phi(r) = \frac{q''r^2}{2kR}$$

و

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}, \text{BC}\begin{cases}\frac{\partial\psi}{\partial r}(R,t) = 0\\ \frac{\partial\psi}{\partial r}(0,t) = 0\end{cases}, \psi(r,0) = -\phi(r) - \varphi(0)$$

$$\psi(r,t) = f(r).g(t) \Rightarrow f(r) = A_{1n}\frac{\sin(\lambda_n r)}{r} + A_{2n}\frac{\cos(\lambda_n r)}{r}$$

$$f'(R) = 0 \Rightarrow A_{2n} = 0, f'(R) = 0$$

$\Rightarrow \lambda_n\cos(\lambda_n R) = \frac{1}{R}\sin(\lambda_n R) \Rightarrow$ λ_n به دست خواهد آمد

$$g(t) = A_{3n}.\exp(-\alpha\lambda_n^2 t)$$

$$\Rightarrow \psi(r,t) = \sum_{n=0}^{\infty} A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\alpha\lambda_n^2 t)$$

$$\psi(r,0) = -\phi(r) - \varphi(0) = -\frac{q''r^2}{2kR} = \sum_{n=0}^{\infty} A_n.\frac{\sin(\lambda_n r)}{r}$$

$$\Rightarrow A_n = \frac{\int_0^R\left(-\frac{q''r^2}{2kR}\right)r\sin(\lambda_n r)dr}{\int_0^R \sin^2(\lambda_n r)dr}$$

$$\theta(r,t) = \sum_{n=0}^{\infty} A_n.\frac{\sin(\lambda_n r)}{r}.\exp(-\alpha\lambda_n^2 t) + \frac{q''r^2}{2kR} + \frac{3q''t}{\rho c_p R}$$

مساله ۲۵-۵)

پروفیل سهموی: $r = c\sqrt{x}$

$$A(x) = \pi r^2 = \pi c^2 x \text{ At } x = 2L, r = R \Rightarrow R = c\sqrt{2L}, c = \frac{R}{\sqrt{2L}}$$

$$A(x) = \frac{\pi R^2}{2L}x, P(x) = \frac{2\pi R}{\sqrt{2L}}\sqrt{x}$$

برای بخش خارج از دیوار:

$$qA|_x - qA|_{x+dx} + q''P(x)dx = \frac{d}{dt}(\rho c A(x)dxT)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right) + \frac{2q''\sqrt{2L}}{k}\sqrt{x} = \frac{1}{\alpha}\frac{\partial}{\partial t}(xT) \Rightarrow \frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x}\right) + m^2\sqrt{x} = \frac{1}{\alpha}x\frac{\partial T}{\partial t}$$

$$\text{BC}\begin{cases}\frac{\partial T}{\partial x}(0,t) = 0\\ T(L,t) = T'(L,t)\end{cases}, \text{IC: } T(x,0) = T_\infty$$

$$\theta = T - T_\infty \Rightarrow \frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x}\right) + m^2\sqrt{x} = \frac{1}{\alpha}x\frac{\partial \theta}{\partial t}$$

$$\text{BC}\begin{cases}\frac{\partial \theta}{\partial x}(0,t) = 0\\ \theta(L,t) = \theta'(L,t)\end{cases}, \text{IC: } \theta(x,0) = 0$$

$$\theta(x,t) = \psi(x,t) + \phi(x)$$

$$\frac{\partial}{\partial x}\left(x\frac{\partial \phi}{\partial x}\right) + m^2\sqrt{x} = 0, \text{BC}\begin{cases}\frac{\partial \phi}{\partial x}(0) = 0\\ \phi(L) = \phi'(L)\end{cases}$$

$$\Rightarrow \phi(x) = C_{1n}J_0\left(\frac{4m}{3}x^{\frac{3}{4}}\right) + C_{2n}Y_0\left(\frac{4m}{3}x^{\frac{3}{4}}\right), \frac{\partial \phi}{\partial x}(0) = 0 \Rightarrow C_{2n} = 0$$

$$\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) + m^2\sqrt{x} = \frac{1}{\alpha}x\frac{\partial \psi}{\partial t}, \text{BC}\begin{cases}\frac{\partial \psi}{\partial x}(0,t) = 0\\ \psi(L,t) = \psi'(L,t)\end{cases}, \text{IC: } \psi(x,0) = -\phi(x)$$

$$\psi(x,t) = f(x).g(t) \Rightarrow f(x) = A_{1n}J_0(\lambda_n x) + A_{2n}Y_0(\lambda_n x)$$

$$\frac{\partial f}{\partial x}(0) = 0 \Rightarrow A_{2n} = 0, \tau(t) = A_{3n}\exp(-\lambda_n^2\alpha t)$$

$$\psi(x,t) = \sum_{n=0}^{\infty} A_n.\exp(-\lambda_n^2\alpha t).J_0(\lambda_n x)$$

$$\psi(x,0) = -\phi(x) = -C_{1n}J_0\left(\frac{4m}{3}x^{\frac{3}{4}}\right) = \sum_{n=0}^{\infty} A_n.J_0(\lambda_n x)$$

$$\Rightarrow A_n = \frac{\int_0^L C_{1n}J_0\left(\frac{4m}{3}x^{\frac{3}{4}}\right).J_0(\lambda_n x)dx}{\int_0^L J_0^2(\lambda_n x)dx}$$

برای بخش درون دیوار:

$$qA|_x - qA|_{x+dx} = \frac{d}{dt}(\rho c A(x)dxT') \to \frac{\partial}{\partial x}\left(x\frac{\partial T'}{\partial x}\right) = \frac{1}{\alpha}x\frac{\partial T'}{\partial t}$$

$$\text{BC}\begin{cases}\frac{\partial T'}{\partial x}(2L,t) = \frac{h}{k}(T' - T_\infty)\\ T(L,t) = T'(L,t)\end{cases}, \text{IC: } T'(x,0) = T_\infty$$

$$\theta' = T' - T_\infty \to \frac{\partial}{\partial x}\left(x\frac{\partial \theta'}{\partial x}\right) = \frac{1}{\alpha}x\frac{\partial \theta'}{\partial t}$$

$$\theta'(x,t) = f'(x).g'(t) \to f'(x) = B_{1n}J_0(\mu_n x) + B_{2n}Y_0(\mu_n x)$$

$$g'(t) = B_{3n}\exp(-\mu_n^2\alpha t)$$

ادامه حل و به دست آوردن ثوابت به خواننده واگذار می‌شود.

مساله ۲۷-۵)

(i) برای ضریب انتقال حرارت بزرگ:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{u''}{k} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \text{BC}\begin{cases}\theta(x,l,t) = 0\\ \frac{\partial \theta}{\partial y}(x,0,t) = 0\\ \theta(L,y,t) = 0\\ \frac{\partial \theta}{\partial x}(0,y,t) = 0\end{cases}, \text{IC: } \theta(x,y,0) = 0$$

$$\theta(x,y,t) = \psi(x,y,t) + \phi(x,y)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{u''}{k} = 0, \text{BC}\begin{cases}\phi(x,l) = 0\\ \frac{\partial \phi}{\partial y}(x,0) = 0\\ \phi(L,y) = 0\\ \frac{\partial \phi}{\partial x}(0,y) = 0\end{cases}$$

$$\phi(x,y) = \varphi(x,y) + \delta(x)$$

$$\frac{\partial^2\delta}{\partial x^2} + \frac{u''}{k} = 0, \text{BC}\begin{cases}\delta(L) = 0\\ \frac{\partial \delta}{\partial x}(0) = 0\end{cases} \Rightarrow \delta(x) = \frac{-u''}{2k}[x^2 - L^2]$$

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0\ ,\ \mathrm{BC}\begin{cases}\varphi(x,l)=-\delta(x)\\ \frac{\partial\varphi}{\partial y}(x,0)=0\\ \varphi(L,y)=-\delta(L)=0\\ \frac{\partial\varphi}{\partial x}(0,y)=0\end{cases}$$

$$\varphi(x,y)=f(x).g(y)\Rightarrow f(x)=C_{1n}\sin(\beta_n x)+C_{2n}\cos(\beta_n x)$$

$$\frac{\partial f}{\partial x}(0)=0\Rightarrow C_{1n}=0\ ,\ f(L)=0\Rightarrow C_{2n}\cos(\beta_n L)=0\Rightarrow \beta_n=\frac{(2n+1)\pi}{2L}$$

$$g(y)=C_{3n}\sinh(\beta_n y)+C_{4n}\cosh(\beta_n y)$$

$$\frac{\partial g}{\partial y}(0)=0\Rightarrow C_{3n}=0\Rightarrow \varphi(x,y)=\sum_{n=0}^{\infty}C_n.\cos(\beta_n x).\cosh(\beta_n y)$$

$$\varphi(x,l)=-\delta(x)=\frac{u''}{2k}[x^2-L^2]=\sum_{n=0}^{\infty}C_n.\cos(\beta_n x).\cosh(\beta_n l)$$

$$\xrightarrow{\times\cos(\beta_n x)dx}C_n=\frac{\int_0^L\frac{u''}{2k}[x^2-L^2].\cos(\beta_n x).dx}{\int_0^L\cos^2(\beta_n x).\cosh(\beta_n l).dx}$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{1}{\alpha}\frac{\partial\psi}{\partial t}\ ,\ \mathrm{BC}\begin{cases}\psi(x,l,t)=0\\ \frac{\partial\psi}{\partial y}(x,0,t)=0\\ \psi(L,y,t)=0\\ \frac{\partial\psi}{\partial x}(0,y,t)=0\end{cases}\ ,\ \mathrm{IC}:\psi(x,y,0)=-\phi(x,y)$$

$$\psi(x,y,t)=X(x).Y(y).\tau(t)$$

$$\frac{X''}{X}=-\frac{Y''}{Y}+\frac{1}{\alpha}\frac{\tau'}{\tau}=-\lambda_n^2\Rightarrow X(x)=A_{1n}\sin(\lambda_n x)+A_{2n}\cos(\lambda_n x)$$

$$\frac{\partial X}{\partial x}(0)=0\Rightarrow A_{1n}=0\ ,X(L)=0\Rightarrow A_{2n}\cos(\lambda_n L)=0\Rightarrow \lambda_n=\frac{(2n+1)\pi}{2L}$$

$$\Rightarrow \lambda_n=\beta_n$$

$$\frac{Y''}{Y}=\frac{1}{\alpha}\frac{\tau'}{\tau}+\lambda_n^2=-\mu_n^2\Rightarrow Y(y)=A_{3n}\sin(\mu_n y)+A_{4n}\cos(\mu_n y)$$

$$\frac{\partial Y}{\partial y}(0)=0\Rightarrow A_{3n}=0, Y(l)=0\Rightarrow A_{4n}\cos(\mu_n l)=0\Rightarrow \mu_n=\frac{(2n+1)\pi}{2l}$$

$$\frac{1}{\alpha}\frac{\tau'}{\tau}=-(\lambda_n^2+\mu_n^2)\Rightarrow \tau(t)=A_{5n}.\exp(-(\lambda_n^2+\mu_n^2)\alpha t)$$

$$\psi(x,y,t)=\sum_{n=0}^{\infty}A_n.\cos(\lambda_n x).\cos(\mu_n y).\exp(-(\lambda_n^2+\mu_n^2)\alpha t)$$

$$\psi(x,y,0)=\sum_{n=0}^{\infty}A_n.\cos(\lambda_n x).\cos(\mu_n y)=-\phi(x,y)=\frac{u''}{2k}[x^2-L^2]-\sum_{n=0}^{\infty}C_n.\cos(\lambda_n x).\cosh(\lambda_n y)$$

یک‌بار در $\cos(\lambda_n x).dx$ ضرب نموده و در بازه صفر تا L انتگرال می‌گیریم و با دیگر در $\cos(\mu_n y).dy$ ضرب نموده و در بازه صفر تا l انتگرال می‌گیریم که نتیجه به صورت زیر خواهد بود:

$$\Rightarrow A_n=\frac{2}{L}\frac{2}{l}\int_0^l\int_0^L(-\phi(x,y)).\cos(\lambda_n x).\cos(\mu_n y).dx.dy$$

با استفاده از این معادله ثابت A_n به دست خواهد آمد که به خواننده واگذار می‌شود.

(ii) برای ضریب انتقال حرارت متوسط:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{u''}{k}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\theta}{\partial y}(x,l,t)=\frac{h}{k}\theta(x,l,t)\\ \frac{\partial\theta}{\partial y}(x,0,t)=0\\ \frac{\partial\theta}{\partial x}(L,y,t)=\frac{h}{k}\theta(L,y,t)\\ \frac{\partial\theta}{\partial x}(0,y,t)=0\end{cases}\ ,\ \mathrm{IC}:\theta(x,y,0)=0$$

$$\theta(x,y,t)=\psi(x,y,t)+\phi(x,y)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{u''}{k}=0\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\phi}{\partial y}(x,l)=0\\ \frac{\partial\phi}{\partial y}(x,0)=0\\ \frac{\partial\phi}{\partial x}(L,y)=0\\ \frac{\partial\phi}{\partial x}(0,y)=0\end{cases}$$

$$\phi(x,y)=\varphi(x,y)+\delta(x)$$

$$\frac{\partial^2\delta}{\partial x^2}+\frac{u''}{k}=0\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\delta}{\partial x}(L)=0\\ \frac{\partial\delta}{\partial x}(0)=0\end{cases}\Rightarrow \delta(x)=\frac{-u''}{2k}[x^2-L^2]$$

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0\ ,\ \mathrm{BC}\begin{cases}\varphi(x,l)=-\delta(x)\\ \frac{\partial\varphi}{\partial y}(x,0)=0\\ \varphi(L,y)=-\delta(L)=0\\ \frac{\partial\varphi}{\partial x}(0,y)=0\end{cases}$$

$\varphi(x,y) = f(x).g(y) \Rightarrow f(x) = C_{1n}\sin(\beta_n x) + C_{2n}\cos(\beta_n x)$

$\frac{\partial f}{\partial x}(0) = 0 \Rightarrow C_{1n} = 0\ ,\ f(L) = 0 \Rightarrow C_{2n}\cos(\beta_n L) = 0 \Rightarrow \beta_n = \frac{(2n+1)\pi}{2L}$

$g(y) = C_{3n}\sinh(\beta_n y) + C_{4n}\cosh(\beta_n y)$

$\frac{\partial g}{\partial y}(0) = 0 \Rightarrow C_{3n} = 0 \Rightarrow \varphi(x,y) = \sum_{n=0}^{\infty} C_n.\cos(\beta_n x).\cosh(\beta_n y)$

$\varphi(x,l) = -\delta(x) = \frac{u''}{2k}[x^2 - L^2] = \sum_{n=0}^{\infty} C_n.\cos(\beta_n x).\cosh(\beta_n l)$

$\xrightarrow{\times\cos(\beta_n x)dx} C_n = \frac{\int_0^L \frac{u''}{2k}[x^2-L^2].\cos(\beta_n x).dx}{\int_0^L \cos^2(\beta_n x).\cosh(\beta_n l).dx}$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}\ ,\ \mathrm{BC}\begin{cases}\frac{\partial\psi}{\partial y}(x,l,t) = \frac{h}{k}\psi(x,l,t)\\ \frac{\partial\psi}{\partial y}(x,0,t) = 0\\ \frac{\partial\psi}{\partial x}(L,y,t) = \frac{h}{k}\psi(L,y,t)\\ \frac{\partial\psi}{\partial x}(0,y,t) = 0\end{cases},$$

$\mathrm{IC}: \psi(x,y,0) = -\phi(x,y)$

$\psi(x,y,t) = X(x).Y(y).\tau(t)$

$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{1}{\alpha}\frac{\tau'}{\tau} = -\lambda_n^2 \Rightarrow X(x) = A_{1n}\sin(\lambda_n x) + A_{2n}\cos(\lambda_n x)$

$\frac{\partial X}{\partial x}(0) = 0 \Rightarrow A_{1n} = 0 \Rightarrow X(x) = A_{2n}\cos(\lambda_n x)\ ,$

$\frac{\partial X}{\partial x}(L) = \frac{h}{k}X(L) \Rightarrow -A_{2n}\lambda_n\sin(\lambda_n L) = \frac{h}{k}A_{2n}\cos(\lambda_n L) \Rightarrow$

λ_n به دست خواهد آمد

$\frac{Y''}{Y} = \frac{1}{\alpha}\frac{\tau'}{\tau} + \lambda_n^2 = -\mu_n^2 \Rightarrow Y(y) = A_{3n}\sin(\mu_n y) + A_{4n}\cos(\mu_n y)$

$\frac{\partial Y}{\partial y}(0) = 0 \Rightarrow A_{3n} = 0 \Rightarrow Y(y) = A_{4n}\cos(\mu_n y),$

$\frac{\partial Y}{\partial y}(l) = \frac{h}{k}Y(l) \Rightarrow -A_{4n}\mu_n\sin(\mu_n l) = \frac{h}{k}\cos(\mu_n L) \Rightarrow$ μ_n به دست خواهد آمد

$\frac{1}{\alpha}\frac{\tau'}{\tau} = -(\lambda_n^2 + \mu_n^2) \Rightarrow \tau(t) = A_{5n}.\exp(-(\lambda_n^2 + \mu_n^2)\alpha t)$

$\psi(x,y,t) = \sum_{n=0}^{\infty} A_n.\cos(\lambda_n x).\cos(\mu_n y).\exp(-(\lambda_n^2 + \mu_n^2)\alpha t)$

$\psi(x,y,0) = \sum_{n=0}^{\infty} A_n.\cos(\lambda_n x).\cos(\mu_n y) = -\phi(x,y) = \frac{u''}{2k}[x^2 - L^2] - \sum_{n=0}^{\infty} C_n.\cos(\lambda_n x).\cosh(\lambda_n y)$

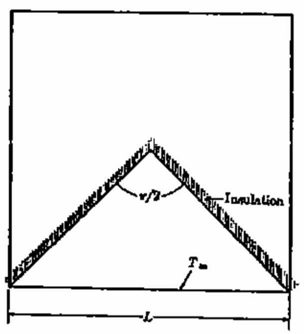
یک‌بار در $\cos(\lambda_n x).dx$ ضرب نموده و در بازه صفر تا L انتگرال می‌گیریم و با دیگر در $\cos(\mu_n y).dy$ ضرب نموده و در بازه صفر تا l انتگرال می‌گیریم که نتیجه به صورت زیر خواهد بود:

$\Rightarrow A_n = \frac{2}{L}\frac{2}{l}\int_0^l\int_0^L(-\phi(x,y)).\cos(\lambda_n x).\cos(\mu_n y).dx.dy$

با استفاده از این معادله ثابت A_n به دست خواهد آمد که به خواننده واگذار می‌شود.

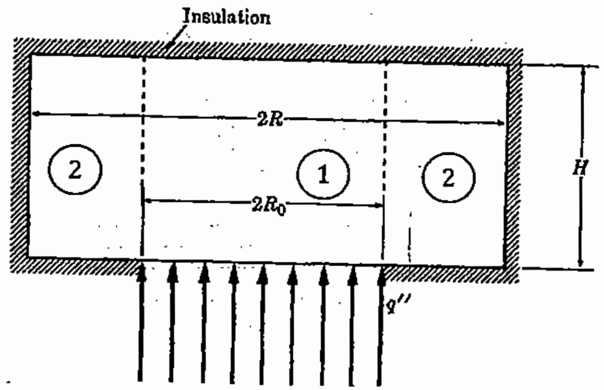
مساله ۲۸-۵)

بله می‌توانیم این مساله را به صورت حاصل‌ضرب دو مساله ناپایای یک بعدی بنویسیم. می‌توانیم مثلث را به صورت یک مربع در نظر بگیریم.



$\left(\frac{T-T_\infty}{T_0-T_\infty}\right)_{2L,2L} = \left(\frac{T-T_\infty}{T_0-T_\infty}\right)_{2L}\cdot\left(\frac{T-T_\infty}{T_0-T_\infty}\right)_{2L} = \left(\frac{T-T_\infty}{T_0-T_\infty}\right)_{2L}^2$

مساله ۲۹-۵)



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right)+\frac{\partial^2 T_1}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T_1}{\partial t}\ ,\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right)+\frac{\partial^2 T_2}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T_2}{\partial t}$$

$$\text{BC}\begin{cases}\frac{\partial T_1}{\partial z}(r,H,t)=0\\ k\frac{\partial T_1}{\partial z}(r,0,t)+q''=0\\ \frac{\partial T_1}{\partial r}(0,z,t)=0\\ T_1(R_o,z,t)=T_2(R_o,z,t)\\ \frac{\partial T_1}{\partial r}(R_o,z,t)=\frac{\partial T_2}{\partial r}(R_o,z,t)\\ \frac{\partial T_2}{\partial r}(R,z,t)=0\\ \frac{\partial T_2}{\partial z}(r,H,t)=0\\ \frac{\partial T_2}{\partial z}(r,0,t)=0\end{cases}\ ,\ \text{IC: } T_1(r,z,0)=T_2(r,z,0)=T_\infty$$

$$\theta=T-T_\infty \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_1}{\partial r}\right)+\frac{\partial^2 \theta_1}{\partial z^2}=\frac{1}{\alpha}\frac{\partial \theta_1}{\partial t}\ ,\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_2}{\partial r}\right)+\frac{\partial^2 \theta_2}{\partial z^2}=\frac{1}{\alpha}\frac{\partial \theta_2}{\partial t}$$

$$\text{BC}\begin{cases}\frac{\partial \theta_1}{\partial z}(r,H,t)=0\\ k\frac{\partial \theta_1}{\partial z}(r,0,t)+q''=0\\ \frac{\partial \theta_1}{\partial r}(0,z,t)=0\\ \theta_1(R_o,z,t)=\theta_2(R_o,z,t)\\ \frac{\partial \theta_1}{\partial r}(R_o,z,t)=\frac{\partial \theta_2}{\partial r}(R_o,z,t)\\ \frac{\partial \theta_2}{\partial r}(R,z,t)=0\\ \frac{\partial \theta_2}{\partial z}(r,H,t)=0\\ \frac{\partial \theta_2}{\partial z}(r,0,t)=0\end{cases}\ ,\ \text{IC: } \theta_1(r,z,0)=\theta_2(r,z,0)=0$$

$$\theta_1(r,z,t)=\psi(r,z,t)+\phi(r,z)+\varphi(z)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{\partial^2 \phi}{\partial z^2}=0\ \text{BC}\begin{cases}\frac{\partial \phi}{\partial z}(r,H)=0\\ k\frac{\partial \phi}{\partial z}(r,0)+q''=0\\ \frac{\partial \phi}{\partial r}(0,z)=0\end{cases}$$

$$\phi(r,z)=R(r).Z(z)\Rightarrow R(r)=C_{1n}J_0(\lambda_n r)+C_{2n}Y_0(\lambda_n r)$$

$$\frac{\partial R}{\partial r}(0)=0\Rightarrow C_{2n}=0$$

$$Z(z)=C_{3n}\sinh(\lambda_n z)+C_{4n}\cosh(\lambda_n z)$$

$$\frac{\partial Z}{\partial z}(H)=0\Rightarrow C_{3n}\lambda_n\cosh(\lambda_n H)-C_{4n}\lambda_n\sinh(\lambda_n H)=0\Rightarrow C_{3n}=C_{4n}\tanh(\lambda_n H)$$

$$\phi(r,z)=\sum_{n=0}^{\infty}C_n.J_0(\lambda_n r).[\sinh(\lambda_n z)\tanh(\lambda_n H)+\cosh(\lambda_n z)]$$

$$\sum_{n=0}^{\infty}C_n.J_0(\lambda_n r).\left[\lambda_n\underbrace{\cosh(\lambda_n.0)}_{1}\tanh(\lambda_n H)+\lambda_n\underbrace{\sinh(\lambda_n.0)}_{0}\right]+\frac{q''}{k}=0$$

$$\xrightarrow{\times rJ_0(\lambda_n r)dr}C_n=\frac{\int_0^{R_o}-\frac{q''}{k}.rJ_0(\lambda_n r)dr}{\int_0^{R_o}rJ_0^2(\lambda_n r)[\lambda_n\tanh(\lambda_n H)]dr}$$

$$\psi(r,z,t)=R_1(r).Z_1(z).\tau_1(t)$$

$$\Rightarrow\psi(r,z,t)=\sum_{n=0}^{\infty}A_nJ_0(\gamma_{1n}r)\cosh(\mu_{1n}z)\exp\left((\mu_{1n}^2-\gamma_{1n}^2)t\right)$$

$-\phi(r,z)=\psi(r,z,0)=\sum_{n=0}^{\infty}A_nJ_0(\gamma r)\cosh(\mu_{1n}z)\Rightarrow$ A_n به دست خواهد آمد

$$\theta_2(r,z,t) = R_2(r).Z_2(z).\tau_2(t)$$

$$\Rightarrow \theta_2(r,z,t) = \sum_{n=0}^{\infty} B_n J_0(\gamma_{2n} r)\cosh(\mu_{2n} z)\exp\left((\mu_{2n}^2 - \gamma_{2n}^2)t\right)$$

ادامه حل و به دست آوردن ثوابت به خواننده واگذار می‌شود.

مساله ۳۰-۵)

$$\mathrm{M}\begin{cases} R = 0 \\ Z = 0 \\ t = 40\ \text{min} \end{cases}, \mathrm{P}\begin{cases} R = 1'' \\ Z = 0 \\ t = 40\ \text{min} \end{cases}$$

$$Bi_R = \frac{hR}{k} = \frac{0\times\frac{1}{12}}{10} = 0\ , Bi_H = \frac{hH}{k} = \frac{30\times\frac{1}{12}}{10} = 0.25, \frac{1}{Bi_R} = \infty\ , \frac{1}{Bi_H} = 4$$

$$\alpha = \frac{10}{500\times 0.1} = 0.2\frac{ft^2}{hr}$$

(a

برای نقطه M :

$$\frac{r}{R} = 0\ , \frac{z}{H} = 0\ , Fo_R = \frac{\alpha t}{R^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2\ , Fo_H = \frac{\alpha t}{H^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2$$

با استفاده از منحنی‌های ارائه شده در فصل ۵ کتاب انتقال حرارت هدایتی آرپاچی شکل ۱۶-۵ و $Fo_R = 19.2$ و $\frac{1}{Bi_R} = \infty$ $(T - T_\infty)/(T_0 - T_\infty) = 1$ برای $\frac{r}{R} = 0$, $\frac{z}{H} = 0$ حاصل می‌شود و به همین ترتیب برای جهت H با استفاده از شکل ۱۴-۵.

$$\left(\frac{\theta}{\theta_0}\right)_{2R} = 1\ , \left(\frac{\theta}{\theta_0}\right)_H = 0.7 \Rightarrow \left(\frac{\theta}{\theta_0}\right)_{2R,H} = \left(\frac{\theta}{\theta_0}\right)_{2R} \times \left(\frac{\theta}{\theta_0}\right)_H$$

$$\Rightarrow \frac{T-200}{50-200} = 1 \times 0.5 \Rightarrow T = 95°\mathrm{F}$$

برای نقطه P :

$$\frac{r}{R} = 1\ , \frac{z}{H} = 0\ , Fo_R = \frac{\alpha t}{R^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2\ , Fo_H = \frac{\alpha t}{H^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2$$

$$\left(\frac{\theta}{\theta_0}\right)_{2R} = 1\ , \left(\frac{\theta}{\theta_0}\right)_H = 0.5 \Rightarrow \left(\frac{\theta}{\theta_0}\right)_{2R,H} = \left(\frac{\theta}{\theta_0}\right)_{2R} \times \left(\frac{\theta}{\theta_0}\right)_H$$

$$\Rightarrow \frac{T-200}{50-200} = 1 \times 0.7 \Rightarrow T = 95°\mathrm{F}$$

(b

برای سیلندر:

$$Fo_R = \frac{\alpha t}{R^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2\ , Bi_{cylender(R)} = \frac{0\times\frac{1}{12}}{10} = 0 \Rightarrow Bi^2 F_o = 0$$

$$\Rightarrow \frac{q_{cylender}}{q_{i,cylender}} = 0 \Rightarrow q_{cylender} = 0$$

برای صفحه:

$$Fo_H = \frac{\alpha t}{H^2} = \frac{0.2\times\frac{40}{60}}{\left(\frac{1}{12}\right)^2} = 19.2\ , Bi_{plate(H)} = \frac{30\times\frac{1}{12}}{10} = 0.25 \Rightarrow Bi^2 F_o = 1.2$$

$$\Rightarrow \frac{q_{plate}}{q_{i,plate}} = 0.98,\quad q_{i,plate} = \pi R^2 L\rho c(T_0 - T_\infty) = -13.6\ Btu \Rightarrow$$

$$q_{plate} = 0.98 \times (-13.6) = -13.4$$

$$\Rightarrow q = q_{plate} + q_{cylender} = -13.4\ But$$

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$$\rho(A2L)c\frac{dT}{dt} - (A2L)u''' = \frac{dE}{dt}, \frac{dE}{dt} = -2Aq_n, q_n = h(T - T_\infty)$$

(a

$$u''' = q_0\left(\frac{t}{t_0}\right)$$

$$\rho(A2L)c\frac{dT}{dt} - (A2L)q_0\left(\frac{t}{t_0}\right) = -2hA(T - T_\infty)$$

$$\Rightarrow \frac{dT}{dt} - \underbrace{\frac{q_0}{\rho c t_0}}_{n} t = -\underbrace{\frac{h}{\rho c L}}_{m}(T - T_\infty) \Rightarrow \frac{dT}{dt} - nt = -m(T - T_\infty)$$

$$\theta = T - T_\infty \Rightarrow \frac{d\theta}{dt} + m\theta - nt = 0,\qquad T(0) = T_\infty \Rightarrow \theta_0 = 0$$

$$\left(\frac{d\theta}{dt} + m\theta = nt\right) \times e^{mt} \Rightarrow (e^{mt}\theta)' = nte^{mt}$$

$$\Rightarrow e^{mt}\theta = n\int t\,e^{mt}dt + C = n\left(-t\frac{e^{mt}}{m} - \frac{e^{mt}}{m^2}\right) + C$$

$$\theta_0 = 0 \Rightarrow C = \frac{n}{m^2} \Rightarrow \theta = \frac{n}{m}\left(t + \frac{1}{m}e^{-mt} - \frac{n}{m}\right)$$

(b

$$u''' = q_0(1 - e^{-bt})$$

$$\frac{dT}{dt} - n(1 - e^{-bt})t = -m(T - T_\infty)$$

$$\theta = T - T_\infty \Rightarrow \frac{d\theta}{dt} - n(1 - e^{-bt})t = -m\theta$$

$$\left(\frac{d\theta}{dt} + m\theta = n(1 - e^{-bt})t\right) \times e^{mt} \Rightarrow (e^{mt}\theta)' = n(1 - e^{-bt})e^{mt}t$$

$$\Rightarrow e^{mt}\theta = n\int(1-e^{-bt})e^{mt}t\,dt + C = n\left(\frac{e^{mt}}{m}-\frac{e^{(m-b)t}}{(m-b)}\right)+C$$

$$\theta_0 = 0 \Rightarrow C = n\left(\frac{1}{(m-b)}-\frac{1}{m}\right)$$

$$\Rightarrow \theta = n\left(\frac{1}{m}-\frac{e^{-bt}}{(m-b)}\right)+n\left(\frac{1}{(m-b)}-\frac{1}{m}\right)e^{-mt}$$

(c

$$u''' = q_0(1+\epsilon\sin(\omega t))$$

$$\frac{dT}{dt}-n(1+\epsilon\sin(\omega t)) = -m(T-T_\infty)$$

$$\theta = T-T_\infty \Rightarrow \frac{d\theta}{dt}-n(1+\epsilon\sin(\omega t)) = -m\theta$$

$$\left(\frac{d\theta}{dt}+m\theta = n(1+\epsilon\sin(\omega t))\right)\times e^{mt} \Rightarrow (e^{mt}\theta)' =$$
$$n(1+\epsilon\sin(\omega t))\,e^{mt}$$

$$\Rightarrow e^{mt}\theta = n\int(1+\epsilon\sin(\omega t))\,e^{mt}\,dt + C$$

$$e^{mt}\theta = n\left[\frac{e^{mt}}{m}+\epsilon\frac{\sin(\omega t).(e^{mt}/m)-(\omega/m^2).\cos(\omega t).e^{mt}}{1+(\omega/m)^2}\right]+C$$

$$\theta_0 = 0 \Rightarrow C = n\left[\epsilon\frac{(\omega/m^2)}{1+(\omega/m)^2}-\frac{1}{m}\right]$$

$$\Rightarrow \theta = n\left[\frac{1}{m}+\epsilon\frac{\sin(\omega t).(1/m)-(\omega/m^2).\cos(\omega t)}{1+(\omega/m)^2}\right]+n\left[\epsilon\frac{(\omega/m^2)}{1+(\omega/m)^2}-\frac{1}{m}\right]e^{-mt}$$